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Mathematical modelling of tall buildings and its foundation under randomly fluctuating wind and earthquake ground motions

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Abstract

In the present paper, a non-dimensional mathematical model for high tower buildings and its foundation under randomly fluctuating wind loads and earthquake ground motions excitations is developed as a nonlinear model to study the system more extensively. The system main equations could be derived using two different derivation methods and linearized in minimal symbolic forms; which facilitate a subsequent numerical simulation in order to investigate the vibration characteristics of whole system. The analysis enables designers to have more insight into the characteristics of high tower buildings of similar configuration but with different geometry and material. The complexity of wind loading with its variations in space and time has been considered. A comprehensive mathematical model of six degrees of freedom is presented and solved for free and forced vibrations.

Keywords: tall building vibrations, modal analysis, foundation vibrations, power spectral density, random wind excitation, earthquake ground motions

List of symbols

C_p	Aerodynamic pressure factor (-)
E	Kinetic energy of the system (J)
E_d	Soil dynamic modulus of elasticity (kp/m^3)
F_{1H}, F_{1V}	Spring and damping forces at C or E in horizontal and vertical direction (kp)
F_{2H}, F_{2V}	Spring and damping forces at D or F in horizontal and vertical direction (kp)
F_{EH}, F_{EV}	Spring and damping forces at s_1 in horizontal and vertical direction due to earthquake effect (kp)
$H(i\Omega)$	imaginary transformation function (-)
J_1, J_2	Mass moment of Inertia of foundation with its accompanied vibrated soil and tall building (kg.s^2.m)
J_F, J_S	Mass moment of Inertia of foundation and accompanied vibrated soil with it (kg.s^2.m)
k_{EH}, k_{EV}	Linear horizontal and vertical equivalent spring stiffness of earth (kp/m)
k_{EK}	Rotational equivalent spring stiffness of earth (kp.m/rad)
k_H, k_V	Linear horizontal and vertical equivalent spring stiffness of building-foundation connection (kp/m)
L	Lagrangian function (-)
m_1, m_2	Total mass of foundation with its accompanied vibrated soil (m_F+ m_S) and tall building (kg)

m_F, m_S Foundation and Vibrating soil mass (kg)
$M_W(t)$ Total turbulent wind moment as a function of time (kp.m)
q_K General coordinates $z_1^*, y_1^*, \varphi_1^*, z_2^*, y_2^*$, and φ_2^* (m, m, rad, m, m, rad)
$R_{Q_iQ_j}(\tau)$ Cross correlation function of the excitations (m^2)
$R_{q_rq_s}(\tau), R_{X_rX_s}(\tau)$ Cross correlation function of response with respect to general and original coordinates (m^2)
$\Re$ Rayleigh's dissipation function(kp.m/s)

r_b Vertical embedding damping constant: the damping constant of radiation (kp.s/m^3)
r_{EK} Rotational equivalent damping coefficient of earth (kp.m.s/rad)
r_{EH}, r_{EV} Linear horizontal and vertical equivalent damping coefficient of earth (kp.s/m)
r_H, r_V Linear horizontal, vertical equivalent damping coefficient of building-foundation connection (kp.s/m)
r_S Damping coefficient of the elastic soil bed (kp.s/m^3)
s_1, s_2 Centre of gravity of the foundation and tall building (-)
$S_{q_iq_i}(\Omega), S_{q_rq_s}(\Omega)$ Auto and cross power spectral density function of response w.r.t. general coordinates (m^2.s/rad)
$S_{Q_iQ_i}(\Omega), S_{Q_rQ_s}(\Omega)$ Auto and cross power spectral density function of excitations (m^2.s/rad)
$S_{X_nX_n}(\Omega), S_{X_rX_s}(\Omega)$ Auto and cross power spectral density function of response w.r.t. original coordinates(m^2.s/rad)
t Time (s)
T_{EK} Spring and damping torques about s_1 in rotational direction (kp.m)
U Potential energy of the system (J)
$\overline{U}(H)$ Average wind velocity along the building height H (m/s)
$U_y(t), U_z(t)$ Random displacement excitation of earthquake in horizontal and vertical direction (m)
$U(z,t)$ Wind speed as a function of space and time (m/s)
$\overline{U}(z)$ Constant part of wind speed as a function of space (m/s)
$U'(z,t)$ Turbulent part of wind speed as a function of space and time (m/s)
V_S Vertical wave velocity (m/s)
$W(t)$ Total turbulent wind force in y^*-direction as a function of time (kp)
$W(z,t)$ Wind load as a function of space and time (kp)
$\overline{W}(z)$ Constant part of wind load as a function of space (kp)
$W'(z,t)$ Turbulent part of wind load as a function of space and time (kp)
$\underline{\hat{x}}$ Amplitude of exponential solution of motion differential equations (m)
$y_o^*(t), z_o^*(t)$ Displacement of point O in the direction of y_o^* and z_o^* – axis (m)
$y_1(\tau), z_1(\tau), \varphi_1(\tau), y_2(\tau), z_2(\tau), \varphi_2(\tau)$ non-dimensional Displacements (-)
$y_1'(\tau), z_1'(\tau), \varphi_1'(\tau), y_2'(\tau), z_2'(\tau), \varphi_2'(\tau)$ non-dimensional velocities (-)
$y_1''(\tau), z_1''(\tau), \varphi_1''(\tau), y_2''(\tau), z_2''(\tau), \varphi_2''(\tau)$ non-dimensional accelerations (-)
$y_1^*(t), z_1^*(t)$ Displacement of gravity centre s_1 of foundation in y_1^* and z_1^* - axis (m)
$y_2^*(t), z_2^*(t)$ Displacement of gravity centre s_2 of high tower building in y_2^* and z_2^* - axis (m)
$y_C^*(t), z_C^*(t)$ Displacement of point C in the direction of y_C^* and z_C^* – axis (m)
$\dot{y}_C^*(t), \dot{z}_C^*(t)$ Velocity of point C in the direction of y_C^* and z_C^* – axis (m/s)
$\ddot{y}_C^*(t), \ddot{z}_C^*(t)$ Acceleration of point C in the direction of y_C^* and z_C^* – axis (m/s^2)
$y_D^*(t), z_D^*(t)$ Displacement of point D in the direction of y_D^* and z_D^* – axis (m)
$\dot{y}_D^*(t), \dot{z}_D^*(t)$ Velocity of point D in the direction of y_D^* and z_D^* – axis (m/s)
$\ddot{y}_D^*(t), \ddot{z}_D^*(t)$ Acceleration of point D in the direction of y_D^* and z_D^* – axis (m/s^2)
$y_E^*(t), z_E^*(t)$ Displacement of point E in the direction of y_E^* and z_E^* – axis (m)

$\dot{y}^*_E(t), \dot{z}^*_E(t)$ Velocity of point E in the direction of y^*_E and z^*_E – axis (m/s)

$\ddot{y}^*_E(t), \ddot{z}^*_E(t)$ Acceleration of point E in the direction of y^*_E and z^*_E – axis (m/s^2)

$y^*_F(t), z^*_F(t)$ Displacement of point F in the direction of y^*_F and z^*_F – axis (m)

$\dot{y}^*_F(t), \dot{z}^*_F(t)$ Velocity of point F in the direction of y^*_F and z^*_F – axis (m/s)

$\ddot{y}^*_F(t), \ddot{z}^*_F(t)$ Acceleration of point F in the direction of y^*_F and z^*_F – axis (m/s^2)

α Profile constant (-)

γ_B Specific weight of the high tower building (kp/m^3)

ρ, ρ_1, and ρ_2 Density of air, foundation, and high tower building respectively (kg/m^3)

τ non-dimensional time [-]

$\varphi^*_o(t)$, $\varphi^*_1(t)$, $\varphi^*_2(t)$ Angular displacements about x^*_o, x^*_1, and x^*_2 – axis [rad]

$\varphi_o(t)$, $\varphi_1(t)$, $\varphi_2(t)$ non-dimensional angular displacement about x^*_o, x^*_1, and x^*_2 – axis [-]

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1. Introduction

Large investments have recently been made for the construction of new medium- and high-rise buildings in the world. In many cases performance-based designs have been the preferred method for these buildings. A main consideration in performance-based seismic design is the estimation of the likely development of structural and nonstructural damage limit-states given a hazard level. For this type of buildings efficient modeling techniques are required able to compute the response at different performance states. Certain structures are less vulnerable against vibration impacts whereas certain others are more vulnerable. As we all know that vibration effects are now cannot be neglected, as our day to day life is affected by them. Study of vibration responses of structures has always been a principal concern for design engineers. Therefore, we do put an eye on the vibrations of buildings and its foundations. Uncontrolled vibration causes devastation. Occurrences of Tsunami, earthquake, collapse of structures are few such most common devastating effects of vibration. Thus the study of vibration responses in advance is of immense importance for sustainable and positive effects of vibrations for the well being of humans.

Nowadays, the new and emerging concept of seismic structural design, the so-called performance-based design, requires careful consideration of all aspects involved in structural analysis. One of the most important aspects of structural analysis is Soil-Structure Interaction (SSI). Such interaction may alter the dynamic characteristics of structures and consequently may be beneficial or detrimental to the performance of structures. Soil conditions at a given site may amplify the response of a structure on a soil deposit. Not taking into account these structural response amplifications may lead to an under-designed structure resulting in a premature collapse during an earthquake. Analytical methods of SSI concentrate mainly on single degree of freedom systems and analysis/design of long and important structures such as large bridges and nuclear power plants, and rarely on regular type buildings. Studies which include SSI effects will help to better predict the performance of structures during future ground motions. State of the art knowledge and analytical approaches require, that, the structure-foundation system to be represented by mathematical models that include the influence of the sub-foundation media.

A research work of Panagiotou, M. (2008) was conducted at University of California San Diego (UCSD) on the seismic design, experimental response, and computational modeling of medium- and high-rise reinforced concrete wall buildings. Kim, S.J. (2008) presented an investigation of the effect of vertical ground motion on reinforced concrete structures studied through a combined analytical-experimental research approach. Krier, D. (2009) analyzed several soil-structure interaction problems. Buildings on elastic foundations were studied and comparisons were made between analytical results and the solutions obtained from a Tera Dysac finite element analysis. Gouasmia, A. et al. (2009) studied the seismic response of an idealized small city composed of five equally spaced, five storey reinforced concrete buildings anchored in a soft soil layer overlaid by a rock half space. These results show response amplification of the buildings in the near field in accordance with the results observed in similar cases. Antonyuk, E.Ya., Timokhin, V.V. (2007) outlined a mathematical model describing the vibrations of buildings and engineering structures with general-type passive shock-absorbers, rigid bodies, and ideal constraints.

Auersch, L. (2008) predicted a practice-oriented environmental building vibrations. A Green's functions method for layered soils is used to build the dynamic stiffness matrix of the soil area that is covered by the foundation. A simple building model is proposed by adding a building mass to the dynamic stiffness of the soil. Belakroum, R., et al. (2008) studied the numerical prediction of the aerodynamic behaviour of rectangular buildings. Simulations were made for rectangles of different side coefficients and different angles of attack. The finite element method is used to simulate fluid flow considered Newtonian and incompressible. Davoodi, M., et al. (2008) used the ambient vibration tests to rely on natural excitations, consequently, it was recommended to perform impulsive test for identifying the hidden dynamic characteristics of the building. Kuźniar, K. and Waszczyszyn, Z. (2006) applied neural networks for computation of fundamental natural periods of buildings with load-bearing walls. The analysis is based on long-term tests performed on actual buildings. The identification problem was formulated as the relation between structural and soil basement parameters, and the fundamental period of building.

Uzdin, A.M. et al. (2009) derived equations for the vibrations of a building on the foundations under consideration. Impossibility of use of traditional methods of the linear-spectral theory for analysis of their earthquake resistance is demonstrated. It is established that the systems under consideration do not possess a natural vibration period, and may have ambiguous solutions for forced vibrations. The influence of city traffic-induced vibration on Vilnius Arch-Cathedral Belfry was investigated (Kliukas, R. et al. 2008). Two sources of dynamic excitation were studied. Conventional city traffic was considered to be a natural source of excitation while excitation imposed artificially by moving a heavily loaded truck was considered to be the source of increased risk excitation. Configuration of equipment on springs is simplified for numerical analysis. A simplified approach and associated equations of motion can be developed to evaluate the response of the equipment with vertical and horizontal forcing functions (Turner, J. 2004). Gong, Y. (2010) developed a free vibration analysis method for space mega frames of super tall buildings. The physical model of a mega frame was idealized as a three-dimensional assemblage of stiffened close-thin-walled tubes with continuously distributed mass and stiffness.

Yang, Y.B. et. al analyzed the wave propagation problems caused by the underground moving trains by the 2.5-dimensional finite/infinite element approach. The near field of the half-space, including the tunnel and parts of the soil, was simulated by finite elements, and the far field extending to infinity by infinite elements. Ground-borne vibrations due to subway trains have sometimes reached the level that cannot be tolerated by residents living in adjacent buildings (Shyu et. al. 2002). Also, approaches for predicting vibrations caused by metro trains moving through the tunnel were developed (Gupta et al. 2007), e.g., a semi-analytical pipe-in-pipe model (Forrest and Hunt 2006a,b) and a coupled periodic finite-element–boundary-element model (Clouteau et al. 2005; Degrande et al. 2006b). Clearly, ground-borne vibrations have become an issue of great concern, which will continuously attract the attention of researchers and engineers worldwide. many research projects on ground-borne vibrations due to subway trains were conducted by field measurement (Vadillo et al. 1996; Degrande et al. 2006a) and empirical or semiempirical prediction models (Kurzweil 1979; Trochides 1991; Melke 1998). These studies provide practical references for solving related problems. However, most of these studies were performed for a specific condition, thereby suffering from the lack of generality. On the other hand, concerning the techniques of simulation, most previous works have been based on the two-dimensional (2D) models (Balendra et al. 1991; Yun et al. 2000; Metrikine and Vrouwenvelder 2000).

Prowell, I. (2011) presented an experimental and numerical investigation into the seismic response of modern wind turbines simultaneously subjected to wind, earthquake, and operational excitation. Ulusoy, H.S. (2011) described a certain class of system identification algorithms with particular emphasis on civil engineering applications. The algorithms originated from system realization theory enabled one to identify finite dimensional, linear, time-invariant models of systems in the state space representation from observed data. Wieser, J. (2011) used OpenSees finite element framework to develop full three dimensional models of four steel moment frame buildings. The incremental dynamic analysis method is employed to evaluate the floor response of inelastic steel moment frame buildings subjected to all three components of a suite of 21 ground motions. Ghafari Oskoei, S.A. (2011) dealt with the dynamic behavior of tall guyed masts under seismic loads. Zhong, P. (2011) utilized a ground motion acceleration time-history as an input to an analytic model of a structure and solved the structural response at each time step of the ground motion record.

Weng, S. (2010) proposed a forward substructuring approach, the eigenproperties of the partitioned substructures were assembled to recover the eigensolutions and eigensensitivities of the global structure, which were tuned to reproduce the experimental measurements through an optimization process. Sonmez, E. (2010) developed semi- active controllers, which were based on real-time estimation of instantaneous (dominant) frequency and the evolutionary power spectral density by time-frequency analysis of either the excitation or the response of the structure. Time-frequency analyses were performed by either short-time Fourier transform or wavelet transform. Soudkhah, M. (2010) examined the dynamic response of surface foundations on sandy soils under both forced and ground motion disturbance. Yao, M.M. (2010) used the direct method for modeling the soil and a tall building together and studied energy transferring from soils to buildings during earthquakes, which is critical for the design of earthquake resistant structures and for upgrading existing structures. Ahearn, E.B. (2010) studied the dynamic effects of wind-induced vibrations on high-mast structures in Laramie, WY, and proposed several retrofits that increase the aerodynamic damping, thereby reducing vibrations.

The ground vibration induced by earthquake ground motions is a complicated dynamic problem due to the involvement of a number of factors along the paths of wave propagation, including the load generation mechanism, the geometry and location of tunnel structures, the irregularity of soil layers, etc. Previously, many research projects on ground-borne vibrations due to earthquakes were conducted by field measurement and empirical or semi-empirical prediction models. These studies provide practical reference for solving related problems. However, most of these studies were performed for a specific condition, thereby suffering from the lack of generality.

Assumptions

1. The high tower building-foundation equivalent system moves only in the y^*- z^* plane.
2. The wind effect is identified as randomly fluctuating wind loads in horizontal direction.
3. $U_y(t)$, $U_Z(t)$ are random ground motions of earthquake in horizontal and vertical directions y and z.

4. The high tower building and its foundation are assumed as rigid bodies.
5. The soil kind under the foundation is assumed as a sandy clay :
   (a) Specific weight $\gamma_B = 1200\,\text{kp/m}^3$.
   (b) Dynamic modulus of elasticity $E_d = 30.6*10^6\ \text{kp/m}^3$.
   (c) Vertical wave velocity (compression direction) $v_S = 500\,\text{m/s}$ (Lorenz, H. 1955).
6. The angular velocities $\varphi_o^*(t), \varphi_1^*(t)$, and $\varphi_2^*(t)$ are very small (<<1).
7. The equivalent spring stiffness k_H, k_{EH}, and k_v are linear.
8. The equivalent damping coefficients r_H, r_{EH}, and r_v are linear.
9. The density of building ρ_2 is taken as 0.1 that of the foundation.
10. The air friction was not considered.
11. The place pressure factor C_p can be replaced through the average load factor of total building.
12. The spectral power density $S_{U_1U_1}(\Omega)$ is independent on the cartesian coordinates z, y.
13. The wind velocity distribution along the height of the building can be described with the equation

$$\overline{U}(z) = \left(\frac{z}{H}\right)^{\alpha} \overline{U}(H).$$

14. The cross spectral power density $S_{U_1U_2}(\Omega)$ can be represented through the coherence spectrum of the wind velocity $U'(z_1,t)$ and $U'(z_2,t)$:

$$\gamma^2_{U_1U_2}(\Omega) = \left|S_{U_1U_2}(\Omega)\right|^2 \Big/ [S_{U_1U_1}(\Omega) \,.\, S_{U_2U_2}(\Omega)]$$

2. Derivation of system equations using D'alembert's principle

The model of the problem to be considered is schematically shown in Fig. 1. This model describing the vibrations of high-tower building and its foundation with general-type equivalent passive springs and dampers, rigid bodies, and some ideal constraints like linear springs and dampers under the effect of randomly fluctuating wind loads and the excitation of earthquake ground motions. In setting up the equations of motion of the equivalent system in Fig. 1, it should be born in mind that the geometric, elastic, and kinetic relations of both high tower building and its foundation must be derived. Moreover the external excitation of wind loads should be prepared.

2.1 Foundation differential equations of motion

Figure 2 shows the free body diagram of foundation with its accompanied vibrating soil.

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2.1.1 Geometric relations of tall building and its foundation

For the linearization of derived equations, let φ_o, φ_1 and φ_2 << 1. Geometric relations of building's foundation are

$z_C^*(t) = z_o^*(t) + 0.5b.\varphi_o^*(t)$, $z_D^*(t) = z_o^*(t) - 0.5b.\varphi_o^*(t)$, $z_E^*(t) = z_1^*(t) + 0.5b.\varphi_1^*(t)$,

$z_F^*(t) = z_1^*(t) - 0.5b.\varphi_1^*(t)$,

$z_2^*(t) = z_o^*(t) - 0.5c.(1 - \cos\varphi_2^*(t)) \approx z_o^*(t)$, $\varphi_2^*(t) = \varphi_o^*(t)$, $y_C^*(t) = y_o^*(t)$,

$y_2^*(t) = y_o^*(t) + 0.5c.\sin\varphi_2^*(t) \approx y_o^*(t) + 0.5c.\varphi_2^*(t)$, $y_D^*(t) = y_o^*(t)$, $y_E^*(t) = y_1^*(t)$, and $y_F^*(t) = y_1^*(t)$

Rearranging the previous geometric relations leads to the following form

$$
\left.\begin{aligned}
&z^*_C(t) = z^*_2(t) + 0.5b.\varphi^*_2(t)\,,\ z^*_D(t) = z^*_2(t) - 0.5b.\varphi^*_2(t)\,,\ z^*_E(t) = z^*_1(t) + 0.5b.\varphi^*_1(t)\,,\\
&z^*_F(t) = z^*_1(t) - 0.5b.\varphi^*_1(t)\\
&y^*_C(t) = y^*_2(t) - 0.5c.\varphi^*_2(t)\,,\ y^*_D(t) = y^*_2(t) - 0.5c.\varphi^*_2(t)\,,\quad y^*_E(t) = y^*_1(t)\text{, and } y^*_F(t) = y^*_1(t)
\end{aligned}\right\} \tag{1}
$$

2.1.2 Elastic relations of building's foundation

Elastic relations of building's foundation have the form

$$
\left.\begin{aligned}
&F_{1V} = k_V.[z^*_E(t) - z^*_C(t)] + r_V.[\dot z^*_E(t) - \dot z^*_C(t)]\,,\ F_{1H} = k_H.[y^*_E(t) - y^*_C(t)] + r_H.[\dot y^*_E(t) - \dot y^*_C(t)]\,,\\
&F_{2V} = k_V.[z^*_F(t) - z^*_D(t)] + r_V.[\dot z^*_F(t) - \dot z^*_D(t)]\,,\quad F_{2H} = k_H.[y^*_F(t) - y^*_D(t)] + r_H.[\dot y^*_F(t) - \dot y^*_D(t)]\,,\\
&F_{EH} = k_{EH}.[y^*_1(t) - U_y(t)] + r_{EH}.[\dot y^*_1(t) - \dot U_y(t)]\,,\ F_{EV} = k_{EV}.[z^*_1(t) - U_z(t)] + r_{EV}.[\dot z^*_1(t) - \dot U_z(t)]\\
&T_{EK} = k_{EK}.\varphi^*_1(t) + r_{EK}.\dot\varphi^*_1(t)
\end{aligned}\right\} \tag{2}
$$

2.1.3 Kinetic relations of building's foundation

Applying Newton's second law for the forces in z- and y-directions and the moments about s_1 results in

$$
\left.\begin{aligned}
&\sum F_z = m_1.\ddot z^*_1(t) = -F_{1V} - F_{2V} - F_{EV}\,,\ \sum F_y = m_1.\ddot y^*_1(t) = -F_{1H} - F_{2H} - F_{EH}\\
&\sum M_{s1} = J_1.\ddot\varphi^*_1(t) = -F_{1V}.0.5b.\cos\varphi^*_1(t) + F_{2V}.0.5b.\cos\varphi^*_1(t) - T_{EK}(t) \approx (F_{2V} - F_{1V}).0.5b - T_{EK}(t)
\end{aligned}\right\} \tag{3}
$$

2.2 Differential equations of motion of high tower building

Figure 3 shows the free body diagram of high tower building with its forces and moments affecting on it.

2.2.1 Aeroelastic relations of wind excitation

Nowadays, the study of the behavior of a structure subjected to hydro or aerodynamic loadings forms an integral part of tasks allocated to engineers. The effect of wind must be taken into consideration during the design phase of tall buildings. The mechanism of wind loads acting on a building is very complex. Substantial works have dealt with this problem. In civil engineering and construction of tall buildings, the assessment of wind loads is required to check the resistance of components of the construction and coating. In recent years, the methods proposed by scientists in this field are constantly being updated. The institutions of global standardization are thus forced each time to review the standards that are in force. Under the effect of wind, a building oscillates according to both directions parallel and perpendicular to the flow and in a torsional mode. Notwithstanding its enormous fascination, wind loading is in fact a parasitic effect, and mostly an obstacle in the way of designing structures for their primary intended use. Without wind, structures – particularly large ones – would probably be a lot easier to design and cheaper.

Dynamic wind pressures acting on buildings are complicated functions of both time and space. The wind load per unit area has the form

$$W(z,t) = C_p.q(z,t) \quad \text{and} \quad q(z,t) = \frac{1}{2}\rho\, U^2(z,t)$$

$$W(z,t) = C_p.\frac{\rho}{2}.U^2(z,t) = C_p.\frac{\rho}{2}.[\overline{U}(z) + U^{'}(z,t)]^2 = C_p.\frac{\rho}{2}.[\overline{U}^2(z) + 2\overline{U}(z).U^{'}(z,t) + U^{'2}(z,t)]$$

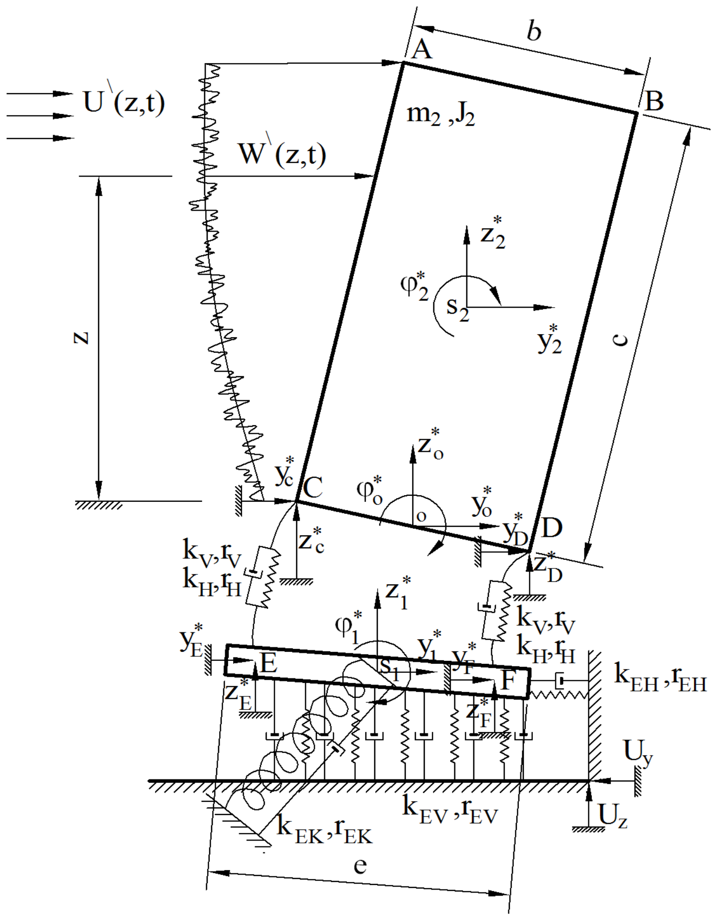

Fig. 1. Equivalent system of tall building and its foundation

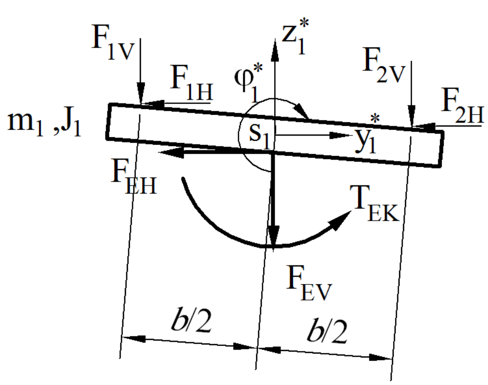

Fig. 2. Free body diagram of foundation with its accompanied vibrated soil

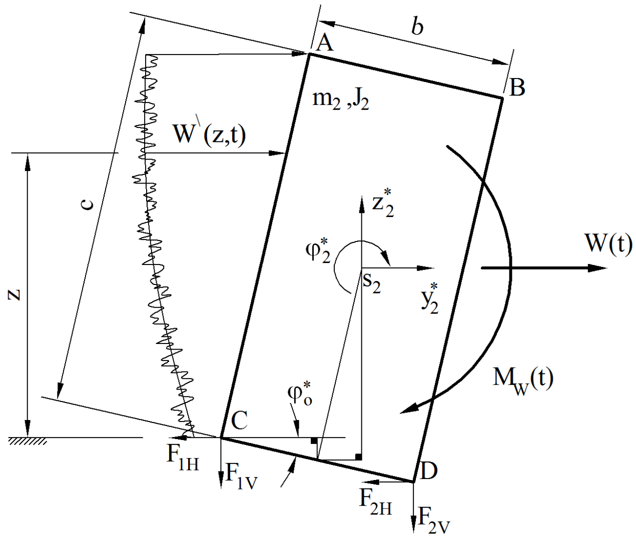

Fig. 3 Free body diagram of the high tower building

$$W(z,t) = C_p.\frac{\rho}{2}.\overline{U}^2(z) + C_p.\rho.\overline{U}(z).U'(z,t) = \overline{W}(z) + W'(z,t)$$

The total turbulent wind force in y^*-direction as a function of time is

$$W(t) = \int_0^c W'(z,t)\,dz = \int_0^c C_p.\rho.\overline{U}(z).U'(z,t)\,dz \tag{4}$$

The total turbulent wind moment as a function of time is

$$M_W(t) = \int_0^c [z - (\frac{c}{2}\cos\varphi_2^*(t) - \frac{b}{2}\sin\varphi_2^*(t))].W'(z,t)\,dz$$

$$\approx \int_0^c (z - \frac{c}{2}).W'(z,t)\,dz = \int_0^c (z - \frac{c}{2}).C_p.\rho.\overline{U}(z).U'(z,t)\,dz \tag{5}$$

2.2.2 Elastic relations of high tower building

Elastic relations of high tower building have the form

$$\left.\begin{aligned} F_{1V} &= k_V.[z_C^*(t) - z_E^*(t)] + r_V.[\dot{z}_C^*(t) - \dot{z}_E^*(t)],\ F_{1H} = k_H.[y_C^*(t) - y_E^*(t)] + r_H.[\dot{y}_C^*(t) - \dot{y}_E^*(t)] \\ F_{2V} &= k_V.[z_D^*(t) - z_F^*(t)] + r_V.[\dot{z}_D^*(t) - \dot{z}_F^*(t)],\ F_{2H} = k_H.[y_D^*(t) - y_F^*(t)] + r_H.[\dot{y}_D^*(t) - \dot{y}_F^*(t)] \end{aligned}\right\} \tag{6}$$

2.2.3 Kinetic relations of high tower building

Applying Newton's second law for the forces in z and y-directions and also the moments about s_2 results in

$$\sum F_z = m_2.\ddot{z}_2^*(t) = -F_{1V} - F_{2V}\ ,\ \sum F_y = m_2.\ddot{y}_2^*(t) = -F_{1H} - F_{2H} + W(t)$$

$$\sum M_{s2} = J_2.\ddot{\varphi}_2^*(t) = -F_{1V}.[\frac{c}{2}.\sin\varphi_2^*(t) + \frac{b}{2}.\cos\varphi_2^*(t)] + F_{1H}.[\frac{c}{2}.\cos\varphi_2^*(t) - \frac{b}{2}.\sin\varphi_2^*(t)]$$

$$+F_{2V}.[-\frac{c}{2}.\mathbf{sin}\varphi_2^*(t)+\frac{b}{2}.\mathbf{cos}\varphi_2^*(t)]\ +F_{2H}.[\frac{c}{2}.\mathbf{cos}\varphi_2^*(t)+\frac{b}{2}.\mathbf{sin}\varphi_2^*(t)]+M_W(t)\ \} \tag{7}$$

The previous equation can be linearized in the following form

$$\sum M_{s2}\approx -F_{1V}.[\frac{b}{2}+\frac{c}{2}.\varphi_2^*(t)]+F_{1H}.[-\frac{b}{2}.\varphi_2^*(t)+\frac{c}{2}]+F_{2V}.[\frac{b}{2}-\frac{c}{2}.\varphi_2^*(t)]+F_{2H}.[\frac{b}{2}.\varphi_2^*(t)+\frac{c}{2}]+M_W(t)$$

2.2.4 Deriving the system's differential equations of motion

2.2.4.1 Application of the geometric relations of the foundation

Substitute from Eqs. 1 in Eqs. 2 of the elastic relations of foundation free body diagram

$$\left.\begin{aligned}
F_{1V}&=k_V.[z_1^*(t)+0.5b.\varphi_1^*(t)-z_2^*(t)-0.5b.\varphi_2^*(t)]+r_V.[\dot z_1^*(t)+0.5b.\dot\varphi_1^*(t)-\dot z_2^*(t)-0.5b.\dot\varphi_2^*(t)]\\
F_{1H}&=k_H.[y_1^*(t)-y_2^*(t)+0.5c\varphi_2^*(t)]+r_H.[\dot y_1^*(t)-\dot y_2^*(t)+0.5c\dot\varphi_2^*(t)]\\
F_{2V}&=k_V.[z_1^*(t)-0.5b.\varphi_1^*(t)-z_2^*(t)+0.5b.\varphi_2^*(t)]+r_V.[\dot z_1^*(t)-0.5b.\dot\varphi_1^*(t)-\dot z_2^*(t)+0.5b.\dot\varphi_2^*(t)]\\
F_{2H}&=k_H.[y_1^*(t)-y_2^*(t)+0.5c\varphi_2^*(t)]+r_H.[\dot y_1^*(t)-\dot y_2^*(t)+0.5c\dot\varphi_2^*(t)]
\end{aligned}\right\} \tag{8}$$

2.2.4.2 Application of the elastic relations of the foundation

Substitute from Eqs. 2 of foundation's elastic relations in Eqs. 3 of its kinetic relations results in

$$\left.\begin{aligned}
m_1.\ddot z_1^*(t)&=-(k_{EV}+2k_V).z_1^*(t)+2k_Vz_2^*(t)-(r_{EV}+2r_V).\dot z_1^*(t)+2r_V.\dot z_2^*(t)+k_{EV}.U_z(t)+r_{EV}\dot U_z(t)\\
m_1.\ddot y_1^*(t)&=-(k_{EH}+2k_H).y_1^*(t)+2k_H.y_2^*(t)-ck_H.\varphi_2^*(t)-(r_{EH}+2r_H).\dot y_1^*(t)\\
&\quad+2r_H.\dot y_2^*(t)-cr_H.\dot\varphi_2^*(t)+k_{EH}.U_y(t)+r_{EH}.\dot U_y(t)\\
J_1.\ddot\varphi_1^*(t)&=-[k_{EK}+0.5b^2.k_V].\varphi_1^*(t)+0.5b^2.k_V.\varphi_2^*(t)-[r_{EK}+0.5b^2.r_V].\dot\varphi_1^*(t)+0.5b^2.r_V.\dot\varphi_2^*(t)
\end{aligned}\right\} \tag{9}$$

2.2.4.3 Application of the geometric relations of the building

Substitute from Eqs. 1 of geometric relations in Eqs. 6 of elastic relations of the building

$$\left.\begin{aligned}
F_{1V}&=k_V.[z_2^*(t)+0.5b.\varphi_2^*(t)-z_1^*(t)-0.5b.\varphi_1^*(t)]+r_V.[\dot z_2^*(t)+0.5b.\dot\varphi_2^*(t)-\dot z_1^*(t)-0.5b.\dot\varphi_1^*(t)]\\
F_{1H}&=-k_H.[y_1^*(t)-y_2^*(t)+0.5c\varphi_2^*(t)]-r_H.[\dot y_1^*(t)-\dot y_2^*(t)+0.5c\dot\varphi_2^*(t)]\\
F_{2V}&=k_V.[z_2^*(t)-0.5b.\varphi_2^*(t)-z_1^*(t)+0.5b.\varphi_1^*(t)]+r_V.[\dot z_2^*(t)-0.5b.\dot\varphi_2^*(t)-\dot z_1^*(t)+0.5b.\dot\varphi_1^*(t)]\\
F_{2H}&=-k_H.[y_1^*(t)-y_2^*(t)+0.5c\varphi_2^*(t)]-r_H.[\dot y_1^*(t)-\dot y_2^*(t)+0.5c\dot\varphi_2^*(t)]
\end{aligned}\right\} \tag{10}$$

2.2.4.4 Application of the elastic relations of the building

Substitute Eqs. 10 of building's elastic relations in Eqs. 7 of its kinetic relations leads to the following differential equations

$$
\left.
\begin{aligned}
&m_2.\ddot{z}_2^*(t) = 2k_V.z_1^*(t) - 2k_V.z_2^*(t) + 2r_V.\dot{z}_1^*(t) - 2r_V.\dot{z}_2^*(t)\\
&m_2.\ddot{y}_2^*(t) = 2k_H.y_1^*(t) - 2k_H.y_2^*(t) + ck_H.\varphi_2^*(t) + 2r_H.\dot{y}_1^*(t) - 2r_H.\dot{y}_2^*(t) + cr_H.\dot{\varphi}_2^*(t)] + W(t)\\
&J_2.\ddot{\varphi}_2^*(t) = -ck_H.y_1^*(t) + 0.5b^2.k_V.\varphi_1^*(t) + ck_H.y_2^*(t) - (0.5c^2.k_H + 0.5b^2.k_V).\varphi_2^*(t)\\
&\qquad - cr_H.\dot{y}_1^*(t) + 0.5b^2.r_V.\dot{\varphi}_1^*(t) + cr_H.\dot{y}_2^*(t) - (0.5c^2.r_H + 0.5b^2.r_V).\dot{\varphi}_2^*(t) + M_W(t)
\end{aligned}
\right\} \quad (11)
$$

2.2.4.5 Arranging the differential equations of motion

The differential equations of motion of both tall building and its foundation can be summarized in the form

$$m_1.\ddot{z}_1^*(t) + (r_{EV} + 2r_V).\dot{z}_1^*(t) - 2r_V.\dot{z}_2^*(t) + (k_{EV} + 2k_V).z_1^*(t) - 2k_V z_2^*(t) = k_{EV}.U_z(t) + r_{EV}\dot{U}_z(t)$$

$$m_1.\ddot{y}_1^*(t) + (r_{EH} + 2r_H).\dot{y}_1^*(t) - 2r_H.\dot{y}_2^*(t) + cr_H.\dot{\varphi}_2^*(t) + (k_{EH} + 2k_H).y_1^*(t) - 2k_H.y_2^*(t) + ck_H.\varphi_2^*(t) = k_{EH}U_y(t) + r_{EH}.\dot{U}_y(t)$$

$$J_1.\ddot{\varphi}_1^*(t) + [r_{EK} + 0.5b^2.r_V].\dot{\varphi}_1^*(t) - 0.5b^2.r_V.\dot{\varphi}_2^*(t) + (k_{EK} + 0.5b^2.k_V).\varphi_1^*(t) - 0.5b^2.k_V.\varphi_2^*(t) = 0$$

$$m_2.\ddot{z}_2^*(t) - 2r_V.\dot{z}_1^*(t) + 2r_V.\dot{z}_2^*(t) - 2k_V.z_1^*(t) + 2k_V.z_2^*(t) = 0$$

$$m_2.\ddot{y}_2^*(t) - 2r_H.\dot{y}_1^*(t) + 2r_H.\dot{y}_2^*(t) - cr_H.\dot{\varphi}_2^*(t) - 2k_H.y_1^*(t) + 2k_H.y_2^*(t) - ck_H.\varphi_2^*(t) = W(t)$$

$$
\begin{aligned}
J_2.\ddot{\varphi}_2^*(t) &+ cr_H.\dot{y}_1^*(t) - 0.5b^2.r_V.\dot{\varphi}_1^*(t) - cr_H.\dot{y}_2^*(t) + (0.5c^2.r_H + 0.5b^2.r_V).\dot{\varphi}_2^*(t)\\
&+ ck_H.y_1^*(t) - 0.5b^2.k_V.\varphi_1^*(t) - ck_H.y_2^*(t) + (0.5c^2.k_H + 0.5b^2.k_V).\varphi_2^*(t) = M_W(t)
\end{aligned}
$$

$$
\begin{bmatrix}
m_1 & 0 & 0 & 0 & 0 & 0\\
0 & m_1 & 0 & 0 & 0 & 0\\
0 & 0 & J_1 & 0 & 0 & 0\\
0 & 0 & 0 & m_2 & 0 & 0\\
0 & 0 & 0 & 0 & m_2 & 0\\
0 & 0 & 0 & 0 & 0 & J_2
\end{bmatrix}
\begin{bmatrix}
\ddot{z}_1^*(t)\\ \ddot{y}_1^*(t)\\ \ddot{\varphi}_1^*(t)\\ \ddot{z}_2^*(t)\\ \ddot{y}_2^*(t)\\ \ddot{\varphi}_2^*(t)
\end{bmatrix}
+
\begin{bmatrix}
r_{EV}+2r_V & 0 & 0 & -2r_V & 0 & 0\\
0 & r_{EH}+2r_H & 0 & 0 & -2r_H & cr_H\\
0 & 0 & r_{EK}+0.5b^2r_V & 0 & 0 & -0.5b^2r_V\\
-2r_V & 0 & 0 & 2r_V & 0 & 0\\
0 & -2r_H & 0 & 0 & 2r_H & -cr_H\\
0 & cr_H & -0.5b^2r_V & 0 & -cr_H & (0.5c^2r_H+0.5b^2r_V)
\end{bmatrix}
\begin{bmatrix}
\dot{z}_1^*(t)\\ \dot{y}_1^*(t)\\ \dot{\varphi}_1^*(t)\\ \dot{z}_2^*(t)\\ \dot{y}_2^*(t)\\ \dot{\varphi}_2^*(t)
\end{bmatrix}
$$

$$
+
\begin{bmatrix}
k_{EV}+2k_V & 0 & 0 & -2k_V & 0 & 0\\
0 & k_{EH}+2k_H & 0 & 0 & -2k_H & ck_H\\
0 & 0 & k_{EK}+0.5b^2k_V & 0 & 0 & -0.5b^2k_V\\
-2k_V & 0 & 0 & 2k_V & 0 & 0\\
0 & -2k_H & 0 & 0 & 2k_H & -ck_H\\
0 & ck_H & -0.5b^2k_V & 0 & -ck_H & (0.5c^2k_H+0.5b^2k_V)
\end{bmatrix}
\begin{bmatrix}
z_1^*(t)\\ y_1^*(t)\\ \varphi_1^*(t)\\ z_2^*(t)\\ y_2^*(t)\\ \varphi_2^*(t)
\end{bmatrix}
=
$$

$$\begin{bmatrix} k_{EV} & r_{EV} & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{EH} & r_{EH} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \dot{\xi}(t) \\ \eta(t) \\ \dot{\eta}(t) \\ W(t) \\ M_W(t) \end{bmatrix}$$

(12)

3 Derivation of system equations using Lagrange's method

The previous obtained system differential equations 12 of motion can be verified using another derivation method, like Lagrange's method using the following Lagrangian Differential Equation

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_K}\right] - \frac{\partial L}{\partial q_K} + \frac{\partial \Re}{\partial \dot{q}_K} = Q_K = \sum_i F_i \frac{\partial \nu_i}{\partial \dot{q}_K} \quad , \quad L = E - U \quad , \quad \Re = \sum_n \frac{1}{2} r_n \nu_n^2 \tag{13}$$

Q_K : General forces, F_i : External forces, and ν_i : Velocity

3.1 Lagrangian function

(a) Kinetic energy of the total equivalent system

$$E = \frac{1}{2} m_1 \dot{z}_1^{*^2}(t) + \frac{1}{2} m_1 \dot{y}_1^{*^2}(t) + \frac{1}{2} J_1 \dot{\varphi}_1^{*^2}(t) + \frac{1}{2} m_2 \dot{z}_2^{*^2}(t) + \frac{1}{2} m_2 \dot{y}_2^{*^2}(t) + \frac{1}{2} J_2 \dot{\varphi}_2^{*^2}(t) \tag{14}$$

(b) Elastic potential energy of the total equivalent system

$$\begin{aligned} U = {} & \frac{1}{2} k_{EV} [z_1^*(t) - U_z(t)]^2 + \frac{1}{2} k_{EH} [y_1^*(t) - U_y(t)]^2 + \frac{1}{2} k_{EK} \varphi_1^{*^2}(t) + \frac{1}{2} k_V [z_E^*(t) - z_C^*(t)]^2 \\ & + \frac{1}{2} k_H [y_E^*(t) - y_C^*(t)]^2 + \frac{1}{2} k_V [z_F^*(t) - z_D^*(t)]^2 + \frac{1}{2} k_H [y_F^*(t) - y_D^*(t)]^2 \end{aligned} \tag{15}$$

(c) Lagrangian function

Using Eqs. 1 and 14-15 to obtain the following Lagrangian function

$$\begin{aligned} L = {} & \frac{1}{2} m_1 \dot{z}_1^{*^2}(t) + \frac{1}{2} m_1 \dot{y}_1^{*^2}(t) + \frac{1}{2} J_1 \dot{\varphi}_1^{*^2}(t) + \frac{1}{2} m_2 \dot{z}_2^{*^2}(t) + \frac{1}{2} m_2 \dot{y}_2^{*^2}(t) + \frac{1}{2} J_2 \dot{\varphi}_2^{*^2}(t) \\ & - \frac{1}{2} k_{EV} [z_1^*(t) - U_z(t)]^2 - \frac{1}{2} k_{EH} [y_1^*(t) - U_y(t)]^2 - \frac{1}{2} k_{EK} \varphi_1^{*^2}(t) \\ & - \frac{1}{2} k_V [z_1^*(t) + 0.5b.\varphi_1^*(t) - z_2^*(t) - 0.5b.\varphi_2^*(t)]^2 - \frac{1}{2} k_H [y_1^*(t) - y_2^*(t) + 0.5c.\varphi_2^*(t)]^2 \\ & - \frac{1}{2} k_V [z_1^*(t) - 0.5b.\varphi_1^*(t) - z_2^*(t) + 0.5b.\varphi_2^*(t)]^2 - \frac{1}{2} k_H [y_1^*(t) - y_2^*(t) + 0.5c.\varphi_2^*(t)]^2 \end{aligned} \tag{16}$$

3.2 Rayleigh's dissipation function

The Rayleigh's dissipation function can be derived as

$$\Re = \sum_{n=1-6} \frac{1}{2} r_6 \nu_6^2 = \frac{1}{2} r_{EV} [\dot{z}_1^*(t) - \dot{U}_z(t)]^2 + \frac{1}{2} r_{EH} [\dot{y}_1^*(t) - \dot{U}_y(t)]^2 + \frac{1}{2} r_{EK} \dot{\varphi}_1^{*2}(t)$$
$$+ \frac{1}{2} r_V [\dot{z}_1^*(t) + 0.5b.\dot{\varphi}_1^*(t) - \dot{z}_2^*(t) - 0.5b.\dot{\varphi}_2^*(t)]^2 + \frac{1}{2} r_H [\dot{y}_1^*(t) - \dot{y}_2^*(t) + 0.5c.\dot{\varphi}_2^*(t)]^2$$
$$+ \frac{1}{2} r_V [\dot{z}_1^*(t) - 0.5b.\dot{\varphi}_1^*(t) - \dot{z}_2^*(t) + 0.5b.\dot{\varphi}_2^*(t)]^2 + \frac{1}{2} r_H [\dot{y}_1^*(t) - \dot{y}_2^*(t) + 0.5c.\dot{\varphi}_2^*(t)]^2 \quad (17)$$

3.3 General external forces

$$Q_K = \sum_i \underline{F}_i \frac{\partial \underline{\nu}_i}{\partial \dot{q}_K} = \sum_i \underline{F}_i \frac{\partial \underline{r}_i}{\partial q_K}, \text{ Where } F_1 = W, F_2 = M_W, \nu_1 = \dot{y}_2^*, \text{ and } \nu_2 = \dot{\varphi}_2^*$$

3.4 Deriving the differential equations of motion

(a) Case of $q_1 = z_1^*(t)$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{z}_1^*(t)}\right] = m_1.\ddot{z}_1^*(t), \quad \frac{\partial L}{\partial z_1^*(t)} = -(k_{EV} + 2k_V) z_1^*(t) + 2k_V.z_2^*(t) + k_{EV}.U_z(t)$$

$$\frac{\partial \Re}{\partial \dot{z}_1^*(t)} = (r_{EV} + 2r_V).\dot{z}_1^*(t) - 2r_V.\dot{z}_2^*(t) - r_{EV}.\dot{U}_z(t), \text{ and } Q_{z_1^*} = 0$$

Substitute from the equations of case (a) in Eq. 13, the first differential equation of motion can be obtained

$$m_1.\ddot{z}_1^*(t) + (r_{EV} + 2r_V).\dot{z}_1^*(t) - 2r_V.\dot{z}_2^*(t) + (k_{EV} + 2k_V).z_1^*(t) - 2k_V z_2^*(t) = k_{EV}.U_z(t) + r_{EV}\dot{U}_z(t) \quad (18)$$

(b) Case of $q_2 = y_1^*(t)$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{y}_1^*(t)}\right] = m_1.\ddot{y}_1^*(t), \quad \frac{\partial L}{\partial y_1^*(t)} = -(2k_H + k_{EH}).y_1^*(t) + 2k_H.y_2^*(t) - ck_H.\varphi_2^*(t) + k_{EH}U_y(t)$$

$$\frac{\partial \Re}{\partial \dot{y}_1^*(t)} = (2r_H + r_{EH}).\dot{y}_1^*(t) - 2r_H.\dot{y}_2^*(t) + cr_H.\dot{\varphi}_2^*(t) - r_{EH}\dot{U}_y(t), \text{ and } Q_{y_1^*} = 0$$

Substitute from the equations of case (b) in Eq. 13, the second differential equation of motion can be obtained

$$m_1.\ddot{y}_1^*(t) + (r_{EH} + 2r_H).\dot{y}_1^*(t) - 2r_H.\dot{y}_2^*(t) + cr_H.\dot{\varphi}_2^*(t) + (k_{EH} + 2k_H).y_1^*(t) - 2k_H.y_2^*(t) + ck_H.\varphi_2^*(t) =$$
$$k_{EH}U_y(t) + r_{EH}.\dot{U}_y(t) \quad (19)$$

(c) Case of $q_3 = \varphi_1^*(t)$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\varphi}_1^*(t)}\right] = J_1.\ddot{\varphi}_1^*(t)\,,\ \frac{\partial L}{\partial \varphi_1^*(t)} = -(k_{EK} + \frac{b^2}{2}k_V).\varphi_1^*(t) + \frac{b^2}{2}k_V.\varphi_2^*(t)$$

$$\frac{\partial \Re}{\partial \dot{\varphi}_1^*(t)} = (r_{EK} + \frac{b^2}{2}r_V).\dot{\varphi}_1^*(t) - \frac{b^2}{2}r_V.\dot{\varphi}_2^*(t)\,, \text{ and } Q_{\dot{\varphi}_1^*} = 0$$

Substitute from the equations of case (c) in Eq. 13, the third differential equation of motion can be obtained

$$J_1.\ddot{\varphi}_1^*(t) + [r_{EK} + 0.5b^2.r_V].\dot{\varphi}_1^*(t) - 0.5b^2.r_V.\dot{\varphi}_2^*(t) + (k_{EK} + 0.5b^2.k_V).\varphi_1^*(t) - 0.5b^2.k_V.\varphi_2^*(t) = 0 \tag{20}$$

(d) Case of $q_4 = z_2^*(t)$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{z}_2^*(t)}\right] = m_2.\ddot{z}_2^*(t)\,,\ \frac{\partial L}{\partial z_2^*(t)} = 2k_V.z_1^*(t) - 2k_V.z_2^*(t)$$

$$\frac{\partial \Re}{\partial \dot{z}_2^*} = -2r_V.\dot{z}_1^*(t) + 2r_V.\dot{z}_2^*(t)\,, \text{ and } Q_{z_2^*} = 0$$

Similarly, the fourth differential equation of motion can be obtained

$$m_2.\ddot{z}_2^*(t) - 2r_V.\dot{z}_1^*(t) + 2r_V.\dot{z}_2^*(t) - 2k_V.z_1^*(t) + 2k_V.z_2^*(t) = 0 \tag{21}$$

(e) Case of $q_5 = y_2^*(t)$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{y}_2^*(t)}\right] = m_2.\ddot{y}_2^*(t)\,,\ \frac{\partial L}{\partial y_2^*(t)} = 2k_H.y_1^*(t) - 2k_H.y_2^*(t) + ck_H.\varphi_2^*(t)$$

$$\frac{\partial \Re}{\partial \dot{y}_2^*(t)} = -2r_H.\dot{y}_1^*(t) + 2r_H.\dot{y}_2^*(t) - cr_H.\dot{\varphi}_2^*(t)\,, \text{ and } Q_{y_2^*} = W(t)$$

Similarly, the fifth differential equation of motion can be obtained

$$m_2.\ddot{y}_2^*(t) - 2r_H.\dot{y}_1^*(t) + 2r_H.\dot{y}_2^*(t) - cr_H.\dot{\varphi}_2^* - 2k_H.y_1^*(t) + 2k_H.y_2^*(t) - ck_H.\varphi_2^* = W(t) \tag{22}$$

(f) Case of $q_6 = \varphi_2^*$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\varphi}_2^*(t)}\right] = J_2.\ddot{\varphi}_2^*(t) \quad , \quad \frac{\partial L}{\partial \varphi_2^*(t)} = -ck_H.y_1^*(t) + \frac{b^2}{2}k_V.\varphi_1^*(t) + ck_H.y_2^*(t) - (\frac{c^2}{2}k_H + \frac{b^2}{2}k_V).\varphi_2^*(t)$$

$$\frac{\partial \Re}{\partial \dot{\varphi}_1^*(t)} = cr_H.\dot{y}_1^*(t) - \frac{b^2}{2}r_V.\dot{\varphi}_1^*(t) - cr_H.\dot{y}_2^*(t) + (\frac{b^2}{2}r_V + \frac{c^2}{2}r_H).\dot{\varphi}_2^*(t)\,, \text{ and } Q_{\varphi_2^*} = M_W(t)$$

Similarly, the sixth differential equation of motion can be obtained

$$J_2.\ddot{\varphi}_2^*(t) + cr_H.\dot{y}_1^*(t) - 0.5b^2.r_V.\dot{\varphi}_1^*(t) - cr_H.\dot{y}_2^*(t) + (0.5c^2.r_H + 0.5b^2.r_V).\dot{\varphi}_2^*(t)$$

$$+ck_H.y_1^*(t) - 0.5b^2.k_V.\varphi_1^*(t) - ck_H.y_2^*(t) + (0.5c^2.k_H + 0.5b^2.k_V).\varphi_2^*(t) = M_W(t) \tag{23}$$

Equations 18-23 can be written in the following matrix form

$$\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}\begin{bmatrix} \ddot{z}_1^*\\ \ddot{y}_1^*\\ \ddot{\varphi}_1^*\\ \ddot{z}_2^*\\ \ddot{y}_2^*\\ \ddot{\varphi}_2^* \end{bmatrix} + \begin{bmatrix} \frac{(r_{EV}+2r_V)}{m_1} & 0 & 0 & -\frac{2r_V}{m_1} & 0 & 0\\ 0 & \frac{r_{EH}+2r_H}{m_1} & 0 & 0 & -\frac{2r_H}{m_1} & \frac{cr_H}{m_1}\\ 0 & 0 & \frac{2r_{EK}+b^2r_V}{2J_1} & 0 & 0 & -\frac{b^2r_V}{2J_1}\\ -\frac{2r_V}{m_2} & 0 & 0 & \frac{2r_V}{m_2} & 0 & 0\\ 0 & -\frac{2r_H}{m_2} & 0 & 0 & \frac{2r_H}{m_2} & -\frac{cr_H}{m_2}\\ 0 & \frac{cr_H}{J_2} & -\frac{b^2r_V}{2J_2} & 0 & -\frac{cr_H}{J_2} & \frac{(c^2r_H+b^2r_V)}{2J_2} \end{bmatrix}\begin{bmatrix} \dot{z}_1^*\\ \dot{y}_1^*\\ \dot{\varphi}_1^*\\ \dot{z}_2^*\\ \dot{y}_2^*\\ \dot{\varphi}_2^* \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{k_{EV}+2k_V}{m_1} & 0 & 0 & -\frac{2k_V}{m_1} & 0 & 0\\ 0 & \frac{k_{EH}+2k_H}{m_1} & 0 & 0 & -\frac{2k_H}{m_1} & \frac{ck_H}{m_1}\\ 0 & 0 & \frac{2k_{EK}+b^2k_V}{2J_1} & 0 & 0 & -\frac{b^2k_V}{2J_1}\\ -\frac{2k_V}{m_2} & 0 & 0 & \frac{2k_V}{m_2} & 0 & 0\\ 0 & -\frac{2k_H}{m_2} & 0 & 0 & \frac{2k_H}{m_2} & -\frac{ck_H}{m_2}\\ 0 & \frac{ck_H}{J_2} & -\frac{b^2k_V}{2J_2} & 0 & -\frac{ck_H}{J_2} & \frac{c^2k_H+b^2k_V}{2J_2} \end{bmatrix}\begin{bmatrix} z_1^*\\ y_1^*\\ \varphi_1^*\\ z_2^*\\ y_2^*\\ \varphi_2^* \end{bmatrix} =$$

$$\begin{bmatrix} \frac{k_{EV}}{m_1} & \frac{r_{EV}}{m_1} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{k_{EH}}{m_1} & \frac{r_{EH}}{m_1} & 0 & 0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&\frac{1}{m_2}&0\\ 0&0&0&0&0&\frac{1}{J_2} \end{bmatrix}\begin{bmatrix} \xi(t)\\ \dot{\xi}(t)\\ \eta(t)\\ \dot{\eta}(t)\\ W(t)\\ M_W(t) \end{bmatrix} \tag{24}$$

3.5 *Normalization of the system differential equations of motion*

The system differential equations of motion of the high tower building with its foundation can be presented in a dimensionless form using the following quantities

$$z_1(t) = \frac{z_1^*(t)}{z_o}, y_1(t) = \frac{y_1^*(t)}{y_o}, \varphi_1(t) = \frac{\varphi_1^*(t)}{\varphi_o}, z_2(t) = \frac{z_2^*(t)}{z_o}, y_2(t) = \frac{y_2^*(t)}{y_o}, \varphi_2(t) = \frac{\varphi_2^*(t)}{\varphi_o}, \xi(t) = \frac{\xi^*(t)}{\xi_o}, \eta(t) = \frac{\eta^*}{\eta_o}$$

where $z_o = y_o = 1$ cm, $\xi_o = \eta_o = 1$ cm and $\varphi_o = 1$ rad .

Applying the time normalization through the following transformations

$\tau=\omega_o t$, $d\tau=\omega_o dt$, where $\omega_o=1$ rad/s and $\dfrac{dz}{dt}=\omega_o\dfrac{dz}{d\tau}$, $\dfrac{d^2z}{dt^2}=\omega_o^2\dfrac{d^2z}{d\tau^2}$, $\Omega_1 t=\dfrac{\Omega_1}{\omega_o}\tau=\eta_1 t$

Therefore the differential equations of motion will be written in the following dimensionless form

$$\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}z_1''(\tau)\\y_1''(\tau)\\\varphi_1''(\tau)\\z_2''(\tau)\\y_2''(\tau)\\\varphi_2''(\tau)\end{bmatrix}+\begin{bmatrix}\frac{(r_{EV}+2r_V)}{m_1\omega_o}&0&0&-\frac{2r_V}{m_1\omega_o}&0&0\\0&\frac{r_{EH}+2r_H}{m_1\omega_o}&0&0&-\frac{2r_H}{m_1\omega_o}&\frac{cr_H}{m_1\omega_o}\\0&0&\frac{2r_{EK}+b^2r_V}{2J_1\omega_o}&0&0&-\frac{b^2r_V}{2J_1\omega_o}\\-\frac{2r_V}{m_2\omega_o}&0&0&\frac{2r_V}{m_2\omega_o}&0&0\\0&-\frac{2r_H}{m_2\omega_o}&0&0&\frac{2r_H}{m_2\omega_o}&-\frac{cr_H}{m_2\omega_o}\\0&\frac{cr_H}{J_2\omega_o}&-\frac{b^2r_V}{2J_2\omega_o}&0&-\frac{cr_H}{J_2\omega_o}&\frac{(c^2r_H+b^2r_V)}{2J_2\omega_o}\end{bmatrix}\begin{bmatrix}z_1'(\tau)\\y_1'(\tau)\\\varphi_1'(\tau)\\z_2'(\tau)\\y_2'(\tau)\\\varphi_2'(\tau)\end{bmatrix}+$$

$$\begin{bmatrix}\frac{k_{EV}+2k_V}{m_1\omega_o^2}&0&0&-\frac{2k_V}{m_1\omega_o^2}&0&0\\0&\frac{k_{EH}+2k_H}{m_1\omega_o^2}&0&0&-\frac{2k_H}{m_1\omega_o^2}&\frac{ck_H\varphi_o}{m_1\omega_o^2y_o}\\0&0&\frac{2k_{EK}+b^2k_V}{2J_1\omega_o^2}&0&0&-\frac{b^2k_V}{2J_1\omega_o^2}\\-\frac{2k_V}{m_2\omega_o^2}&0&0&\frac{2k_V}{m_2\omega_o^2}&0&0\\0&-\frac{2k_H}{m_2\omega_o^2}&0&0&\frac{2k_H}{m_2\omega_o^2}&-\frac{ck_H\varphi_o}{m_2\omega_o^2y_o}\\0&\frac{ck_Hy_o}{J_2\omega_o^2\varphi_o}&-\frac{b^2k_V}{2J_2\omega_o^2}&0&-\frac{ck_Hy_o}{J_2\omega_o^2\varphi_o}&\frac{c^2k_H+b^2k_V}{2J_2\omega_o^2}\end{bmatrix}\begin{bmatrix}z_1(\tau)\\y_1(\tau)\\\varphi_1(\tau)\\z_2(\tau)\\y_2(\tau)\\\varphi_2(\tau)\end{bmatrix}=$$

$$\begin{bmatrix}\frac{k_{EV}}{m_1\omega_o^2z_o}&\frac{r_{EV}}{m_1\omega_oz_o}&0&0&0&0\\0&0&\frac{k_{EH}}{m_1\omega_o^2y_o}&\frac{r_{EH}}{m_1\omega_oy_o}&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&\frac{1}{m_2\omega_o^2y_o}&0\\0&0&0&0&0&\frac{1}{J_2\omega_o^2\varphi_o}\end{bmatrix}\begin{bmatrix}\xi(\tau)\\\dot\xi(\tau)\\\eta(\tau)\\\dot\eta(\tau)\\W(\tau)\\M_W(\tau)\end{bmatrix}\tag{25}$$

4 Analytical solutions using the general modal analysis method

4.1 Eigen value problem

4.1.1 Homogeneous differential equations without damping

$$\underline{M}^{*}\,\underline{\ddot{x}}^{*}(t)+\underline{K}^{*}\,\underline{x}^{*}(t)=\underline{0} \tag{26}$$

Assume that the exponential solutions of Eqs. 26 have the form

$$\underline{x}^{*}(t)=\underline{\hat{x}}\,e^{i\omega t} \tag{27}$$

Applying the solutions of Eqs. 27 in Eqs. 26 leads to the general eigen value problem

$$(-\omega^{2}\underline{M}^{*}+\underline{K}^{*})\,\underline{\hat{x}}\,e^{i\omega t}=\underline{0} \qquad \text{or} \qquad (\underline{A}-\omega^{2}\underline{I})\,\underline{\hat{x}}=\underline{0}$$

Where the matrix $\underline{A}$ has the form

$$\underline{A}=\underline{M}^{*-1}.\underline{K}^{*}=\begin{bmatrix} \frac{k_{EV}+2k_V}{m_1} & 0 & 0 & -\frac{2k_V}{m_1} & 0 & 0 \\ 0 & \frac{k_{EH}+2k_H}{m_1} & 0 & 0 & -\frac{2k_H}{m_1} & \frac{ck_H}{m_1} \\ 0 & 0 & \frac{2k_{EK}+b^2k_V}{2J_1} & 0 & 0 & -\frac{b^2k_V}{2J_1} \\ -\frac{2k_V}{m_2} & 0 & 0 & \frac{2k_V}{m_2} & 0 & 0 \\ 0 & -\frac{2k_H}{m_2} & 0 & 0 & \frac{2k_H}{m_2} & -\frac{ck_H}{m_2} \\ 0 & \frac{ck_H}{J_2} & -\frac{b^2k_V}{2J_2} & 0 & -\frac{ck_H}{J_2} & \frac{c^2k_H+b^2k_V}{2J_2} \end{bmatrix} \tag{28}$$

Using equation 3 one can obtain 12 eigen values $(\pm\omega_1,\pm\omega_2,\pm\omega_3,\pm\omega_4,\pm\omega_5,\pm\omega_6)$ and 6 eigen vectors $(\underline{\hat{x}}_1,\underline{\hat{x}}_2,\underline{\hat{x}}_3,\underline{\hat{x}}_4,\underline{\hat{x}}_5,\underline{\hat{x}}_6)$.

4.2 Modal matrix

The modal matrix has the form

$$\underline{\chi} = [\underline{\hat{x}}_1, \underline{\hat{x}}_2, \underline{\hat{x}}_3, \underline{\hat{x}}_4, \underline{\hat{x}}_5, \underline{\hat{x}}_6] = \begin{bmatrix} \hat{\chi}_{11} & \hat{\chi}_{12} & \hat{\chi}_{13} & \hat{\chi}_{14} & \hat{\chi}_{15} & \hat{\chi}_{16} \\ \hat{\chi}_{21} & \hat{\chi}_{22} & \hat{\chi}_{23} & \hat{\chi}_{24} & \hat{\chi}_{25} & \hat{\chi}_{26} \\ \hat{\chi}_{31} & \hat{\chi}_{32} & \hat{\chi}_{33} & \hat{\chi}_{34} & \hat{\chi}_{35} & \hat{\chi}_{36} \\ \hat{\chi}_{41} & \hat{\chi}_{42} & \hat{\chi}_{43} & \hat{\chi}_{44} & \hat{\chi}_{45} & \hat{\chi}_{46} \\ \hat{\chi}_{51} & \hat{\chi}_{52} & \hat{\chi}_{53} & \hat{\chi}_{54} & \hat{\chi}_{55} & \hat{\chi}_{56} \\ \hat{\chi}_{61} & \hat{\chi}_{62} & \hat{\chi}_{63} & \hat{\chi}_{64} & \hat{\chi}_{65} & \hat{\chi}_{66} \end{bmatrix} ,$$

$$\underline{\chi}^T = \begin{bmatrix} \underline{\hat{x}}_1^T \\ \underline{\hat{x}}_2^T \\ \underline{\hat{x}}_3^T \\ \underline{\hat{x}}_4^T \\ \underline{\hat{x}}_5^T \\ \underline{\hat{x}}_6^T \end{bmatrix} = \begin{bmatrix} \hat{\chi}_{11} & \hat{\chi}_{21} & \hat{\chi}_{31} & \hat{\chi}_{41} & \hat{\chi}_{51} & \hat{\chi}_{61} \\ \hat{\chi}_{12} & \hat{\chi}_{22} & \hat{\chi}_{32} & \hat{\chi}_{42} & \hat{\chi}_{52} & \hat{\chi}_{62} \\ \hat{\chi}_{13} & \hat{\chi}_{23} & \hat{\chi}_{33} & \hat{\chi}_{43} & \hat{\chi}_{53} & \hat{\chi}_{63} \\ \hat{\chi}_{14} & \hat{\chi}_{24} & \hat{\chi}_{34} & \hat{\chi}_{44} & \hat{\chi}_{54} & \hat{\chi}_{64} \\ \hat{\chi}_{15} & \hat{\chi}_{25} & \hat{\chi}_{35} & \hat{\chi}_{45} & \hat{\chi}_{55} & \hat{\chi}_{65} \\ \hat{\chi}_{16} & \hat{\chi}_{26} & \hat{\chi}_{36} & \hat{\chi}_{46} & \hat{\chi}_{56} & \hat{\chi}_{66} \end{bmatrix}$$

4.3 Decoupling of the system differential equations

The transformation of coordinates can be carried out using the equation

$$\underline{x}^{*} = \underline{\chi} \cdot \underline{q}$$

and the system of the vibration differential equations will has the form

$$\underline{\chi}^T \underline{M}^* \underline{\chi}\, \underline{\ddot{q}} + \underline{\chi}^T \underline{R}^* \underline{\chi}\, \underline{\dot{q}} + \underline{\chi}^T \underline{K}^* \underline{\chi}\, \underline{q} = \underline{\chi}^T \underline{B}^* \underline{U}^*$$

Where $\underline{\chi}^T \underline{M}^* \underline{\chi} = \underline{I}$, $\underline{\chi}^T \underline{R}^* \underline{\chi} \approx \text{diag.}[2\,D\omega]$, $\underline{\chi}^T \underline{K}^* \underline{\chi} \approx \text{diag.}[\omega^2]$, and $\underline{\chi}^T \underline{F}^*(t) \approx \text{diag.}[\omega^2]\,\underline{Q}$

When the damping forces of the equivalent system are smaller than its elastic restoring forces, then the coupled terms of the transformed damping matrix can be neglected without any great error. The decoupled differential equations of the system will have the form

$$\underline{I}\,\underline{\ddot{q}} + \text{diag.}[2D\omega]\,\underline{\dot{q}} + \text{diag.}[\omega^2]\,\underline{q} = \text{diag.}[\omega^2]\,\underline{Q}$$

$$\ddot{q}_n(t) + 2D_n\omega_n\,\dot{q}_n(t) + \omega_n^2\,q_n(t) = \omega_n^2\,Q_n(t)\,, \qquad n = 1, 2, \ldots, 6$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \\ \ddot{q}_3(t) \\ \ddot{q}_4(t) \\ \ddot{q}_5(t) \\ \ddot{q}_6(t) \end{bmatrix} + \begin{bmatrix} 2D_1\omega_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2D_2\omega_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2D_3\omega_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2D_4\omega_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2D_5\omega_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2D_6\omega_6 \end{bmatrix} \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \\ \dot{q}_4(t) \\ \dot{q}_5(t) \\ \dot{q}_6(t) \end{bmatrix} +$$

$$\begin{bmatrix} \omega_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_2^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_4^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_5^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_6^2 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \\ q_5(t) \\ q_6(t) \end{bmatrix} = \begin{bmatrix} \omega_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_2^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_4^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_5^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_6^2 \end{bmatrix} \begin{bmatrix} Q_1(t) \\ Q_2(t) \\ Q_3(t) \\ Q_4(t) \\ Q_5(t) \\ Q_6(t) \end{bmatrix} \tag{29}$$

The general external excitations of the system are

$$\underline{Q}(t) = \text{diag.}[\omega^2]^{-1}.\underline{\chi}^T\, \underline{B}^*\, \underline{U}^*(t)$$

$$= \begin{bmatrix} 1/\omega_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\omega_2^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\omega_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\omega_4^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\omega_5^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\omega_6^2 \end{bmatrix} \begin{bmatrix} \widehat{\chi}_{11} & \widehat{\chi}_{21} & \widehat{\chi}_{31} & \widehat{\chi}_{41} & \widehat{\chi}_{51} & \widehat{\chi}_{61} \\ \widehat{\chi}_{12} & \widehat{\chi}_{22} & \widehat{\chi}_{32} & \widehat{\chi}_{42} & \widehat{\chi}_{52} & \widehat{\chi}_{62} \\ \widehat{\chi}_{13} & \widehat{\chi}_{23} & \widehat{\chi}_{33} & \widehat{\chi}_{43} & \widehat{\chi}_{53} & \widehat{\chi}_{63} \\ \widehat{\chi}_{14} & \widehat{\chi}_{24} & \widehat{\chi}_{34} & \widehat{\chi}_{44} & \widehat{\chi}_{54} & \widehat{\chi}_{64} \\ \widehat{\chi}_{15} & \widehat{\chi}_{25} & \widehat{\chi}_{35} & \widehat{\chi}_{45} & \widehat{\chi}_{55} & \widehat{\chi}_{65} \\ \widehat{\chi}_{16} & \widehat{\chi}_{26} & \widehat{\chi}_{36} & \widehat{\chi}_{46} & \widehat{\chi}_{56} & \widehat{\chi}_{66} \end{bmatrix} \begin{bmatrix} k_{EV} & r_{EV} & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{EH} & r_{EH} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \dot{\xi}(t) \\ \eta(t) \\ \dot{\eta}(t) \\ W(t) \\ M_W(t) \end{bmatrix}$$

$$\underline{Q}(t) = \text{diag.}[\frac{1}{\omega^2}].\underline{\chi}^T.\underline{F}^*(t) \qquad \text{and} \quad Q_n(t) = \frac{1}{\omega_n^2}.\chi_n^T.F_n(t), \qquad n = 1, 2, \ldots, 6$$

$$= \begin{bmatrix} (1/\omega_1^2)\widehat{\chi}_{11} & (1/\omega_1^2)\widehat{\chi}_{21} & (1/\omega_1^2)\widehat{\chi}_{31} & (1/\omega_1^2)\widehat{\chi}_{41} & (1/\omega_1^2)\widehat{\chi}_{51} & (1/\omega_1^2)\widehat{\chi}_{61} \\ (1/\omega_2^2)\widehat{\chi}_{12} & (1/\omega_2^2)\widehat{\chi}_{22} & (1/\omega_2^2)\widehat{\chi}_{32} & (1/\omega_2^2)\widehat{\chi}_{42} & (1/\omega_2^2)\widehat{\chi}_{52} & (1/\omega_2^2)\widehat{\chi}_{62} \\ (1/\omega_3^2)\widehat{\chi}_{13} & (1/\omega_3^2)\widehat{\chi}_{23} & (1/\omega_3^2)\widehat{\chi}_{33} & (1/\omega_3^2)\widehat{\chi}_{43} & (1/\omega_3^2)\widehat{\chi}_{53} & (1/\omega_3^2)\widehat{\chi}_{63} \\ (1/\omega_4^2)\widehat{\chi}_{14} & (1/\omega_4^2)\widehat{\chi}_{24} & (1/\omega_4^2)\widehat{\chi}_{34} & (1/\omega_4^2)\widehat{\chi}_{44} & (1/\omega_4^2)\widehat{\chi}_{54} & (1/\omega_4^2)\widehat{\chi}_{64} \\ (1/\omega_5^2)\widehat{\chi}_{15} & (1/\omega_5^2)\widehat{\chi}_{25} & (1/\omega_5^2)\widehat{\chi}_{35} & (1/\omega_5^2)\widehat{\chi}_{45} & (1/\omega_5^2)\widehat{\chi}_{55} & (1/\omega_5^2)\widehat{\chi}_{65} \\ (1/\omega_6^2)\widehat{\chi}_{16} & (1/\omega_6^2)\widehat{\chi}_{26} & (1/\omega_6^2)\widehat{\chi}_{36} & (1/\omega_6^2)\widehat{\chi}_{46} & (1/\omega_6^2)\widehat{\chi}_{56} & (1/\omega_6^2)\widehat{\chi}_{66} \end{bmatrix} \begin{bmatrix} k_{EV} & r_{EV} & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{EH} & r_{EH} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \dot{\xi}(t) \\ \eta(t) \\ \dot{\eta}(t) \\ W(t) \\ M_W(t) \end{bmatrix}$$

$$Q_n(t) = \begin{bmatrix} (1/\omega_1^2)\widehat{\chi}_{11}.k_{EV} & (1/\omega_1^2)\widehat{\chi}_{11}.r_{EV} & (1/\omega_1^2)\widehat{\chi}_{21}.k_{EH} & (1/\omega_1^2)\widehat{\chi}_{21}.r_{EH} & (1/\omega_1^2)\widehat{\chi}_{51}.1 & (1/\omega_1^2)\widehat{\chi}_{61}.1 \\ (1/\omega_2^2)\widehat{\chi}_{12}.k_{EV} & (1/\omega_2^2)\widehat{\chi}_{12}.r_{EV} & (1/\omega_2^2)\widehat{\chi}_{22}.k_{EH} & (1/\omega_2^2)\widehat{\chi}_{22}.r_{EH} & (1/\omega_2^2)\widehat{\chi}_{52}.1 & (1/\omega_2^2)\widehat{\chi}_{62}.1 \\ (1/\omega_3^2)\widehat{\chi}_{13}.k_{EV} & (1/\omega_3^2)\widehat{\chi}_{13}.r_{EV} & (1/\omega_3^2)\widehat{\chi}_{23}.k_{EH} & (1/\omega_3^2)\widehat{\chi}_{23}.r_{EH} & (1/\omega_3^2)\widehat{\chi}_{53}.1 & (1/\omega_3^2)\widehat{\chi}_{63}.1 \\ (1/\omega_4^2)\widehat{\chi}_{14}.k_{EV} & (1/\omega_4^2)\widehat{\chi}_{14}.r_{EV} & (1/\omega_4^2)\widehat{\chi}_{24}.k_{EH} & (1/\omega_4^2)\widehat{\chi}_{24}.r_{EH} & (1/\omega_4^2)\widehat{\chi}_{54}.1 & (1/\omega_4^2)\widehat{\chi}_{64}.1 \\ (1/\omega_5^2)\widehat{\chi}_{15}.k_{EV} & (1/\omega_5^2)\widehat{\chi}_{15}.r_{EV} & (1/\omega_5^2)\widehat{\chi}_{25}.k_{EH} & (1/\omega_5^2)\widehat{\chi}_{25}.r_{EH} & (1/\omega_5^2)\widehat{\chi}_{55}.1 & (1/\omega_5^2)\widehat{\chi}_{65}.1 \\ (1/\omega_6^2)\widehat{\chi}_{16}.k_{EV} & (1/\omega_6^2)\widehat{\chi}_{16}.r_{EV} & (1/\omega_6^2)\widehat{\chi}_{26}.k_{EH} & (1/\omega_6^2)\widehat{\chi}_{26}.r_{EH} & (1/\omega_6^2)\widehat{\chi}_{56}.1 & (1/\omega_6^2)\widehat{\chi}_{66}.1 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \dot{\xi}(t) \\ \eta(t) \\ \dot{\eta}(t) \\ W(t) \\ M_W(t) \end{bmatrix} \tag{30}$$

$$Q_n(t) = \frac{1}{\omega_n^2}[B_{n1}\xi(t) + B_{n2}\dot{\xi}(t) + B_{n3}\eta(t) + B_{n4}\dot{\eta}(t) + B_{n5}W(t) + B_{n6}M_W(t)]$$

Applying the total turbulent wind forces W(t) in y-direction and the total wind moments $M_W(t)$ on the previous equations.

$$Q_n(t) = \frac{1}{\omega_n^2}[B_{n1}\xi(t) + B_{n2}\dot{\xi}(t) + B_{n3}\eta(t) + B_{n4}\dot{\eta}(t) + B_{n5}\int^{A} C_p.\rho.\overline{U}(z).U'(z,t)\,dA + B_{n6}\int^{A}(z - \frac{c}{2}).C_p.\rho.\overline{U}(z).U'(z,t)\,dA] \tag{31}$$

The decoupled system of differential equations can be presented in the following form

$$m_1\ddot{q}_1(t) + r_1\dot{q}_1(t) + k_1q_1(t) = \chi_{11}f_1(t) + \chi_{21}f_2(t) + \chi_{31}f_3(t) + \chi_{41}f_4(t) + \chi_{51}f_5(t) + \chi_{61}f_6(t)$$

$$m_2\ddot{q}_2(t) + r_2\dot{q}_2(t) + k_2q_2(t) = \chi_{12}f_1(t) + \chi_{22}f_2(t) + \chi_{32}f_3(t) + \chi_{42}f_4(t) + \chi_{52}f_5(t) + \chi_{62}f_6(t)$$

$$\left.\begin{aligned}
&m_3\ddot{q}_3(t)+r_3\dot{q}_3(t)+k_3q_3(t)=\chi_{13}f_1(t)+\chi_{23}f_2(t)+\chi_{33}f_3(t)+\chi_{43}f_4(t)+\chi_{53}f_5(t)+\chi_{63}f_6(t)\\
&m_4\ddot{q}_4(t)+r_4\dot{q}_4(t)+k_4q_4(t)=\chi_{14}f_1(t)+\chi_{24}f_2(t)+\chi_{34}f_3(t)+\chi_{44}f_4(t)+\chi_{54}f_5(t)+\chi_{64}f_6(t)\\
&m_5\ddot{q}_5(t)+r_5\dot{q}_5(t)+k_5q_5(t)=\chi_{15}f_1(t)+\chi_{25}f_2(t)+\chi_{35}f_3(t)+\chi_{45}f_4(t)+\chi_{55}f_5(t)+\chi_{65}f_6(t)\\
&m_6\ddot{q}_6(t)+r_6\dot{q}_6(t)+k_6q_6(t)=\chi_{16}f_1(t)+\chi_{26}f_2(t)+\chi_{36}f_3(t)+\chi_{46}f_4(t)+\chi_{56}f_5(t)+\chi_{66}f_6(t)
\end{aligned}\right\} \tag{32}$$

$$\ddot{q}_i(t)+(\frac{r_i}{m_i})\dot{q}_i(t)+(\frac{k_i}{m_i})q_i(t)=(\frac{\chi_{1i}}{m_i})[k_{EV}\xi(t)+r_{EV}\dot{\xi}(t)]+(\frac{\chi_{2i}}{m_i})[k_{EH}\eta(t)+r_{EH}\dot{\eta}(t)]+(\frac{\chi_{3i}}{m_i})f_3(t)+ (\frac{\chi_{4i}}{m_i})f_4(t)+(\frac{\chi_{5i}}{m_i})f_5(t)+(\frac{\chi_{6i}}{m_i})f_6(t) \tag{33}$$

From the previous equations, one can obtain the following imaginary transformation functions

$$\left.\begin{aligned}
&H_1(\Omega)=\frac{\frac{\chi_{1i}}{m_i}[k_{EV}+j\,r_{EV}\Omega]}{(-\Omega^2+\omega_i^2)+j(2D_i\omega_i\Omega)}=\frac{\frac{\chi_{1i}}{m_i}.\frac{1}{\omega_i^2}[k_{EV}+j\,r_{EV}\Omega]}{[1-(\frac{\Omega}{\omega_i})^2]+j(2D_i\frac{\Omega}{\omega_i})}=\frac{\frac{\chi_{1i}}{k_i}[k_{EV}+j\,r_{EV}\Omega]}{[1-(\frac{\Omega}{\omega_i})^2]+j(2D_i\frac{\Omega}{\omega_i})}\\
&H_2(\Omega)=\frac{\frac{\chi_{2i}}{k_i}[k_{EH}+j\,r_{EH}\Omega]}{[1-(\frac{\Omega}{\omega_i})^2]+j(2D_i\frac{\Omega}{\omega_i})},\quad H_3(\Omega)=\frac{\frac{\chi_{3i}}{k_i}.1}{[1-(\frac{\Omega}{\omega_i})^2]+j(2D_i\frac{\Omega}{\omega_i})},\\
&H_4(\Omega)=\frac{\frac{\chi_{4i}}{k_i}.1}{[1-(\frac{\Omega}{\omega_i})^2]+j(2D_i\frac{\Omega}{\omega_i})}\\
&H_5(\Omega)=\frac{\frac{\chi_{5i}}{k_i}.1}{[1-(\frac{\Omega}{\omega_i})^2]+j(2D_i\frac{\Omega}{\omega_i})},\quad H_6(\Omega)=\frac{\frac{\chi_{6i}}{k_i}.1}{[1-(\frac{\Omega}{\omega_i})^2]+j(2D_i\frac{\Omega}{\omega_i})},
\end{aligned}\right\} \tag{34}$$

where $\frac{r_i}{m_i}=2D_i\omega_i$ and $\frac{k_i}{m_i}=\omega_i^2$

A dynamical system with known properties responds to a dynamical loading in a known manner, provided the time-description of the loading is available a priori. Such description is however not possible in case of the excitations due to earthquake ground motions or fluctuating wind loads. Therefore, the safety of a structural system has to be ensured by stochastic modeling of these motions for perceived seismic hazard at the site of the system and by predicting the structural response in probabilistic sense with the help of well-known concepts of random vibration theory. This theory estimates the statistical variations in the peak structural response due to possible variations in the time-description of the excitation (there may be several `different looking' time-histories corresponding to a given characterization of the excitation). The classical random vibration theory makes use of the frequency distribution of input energy as obtained from the Fourier Transform of the excitation. However, since Fourier Transform gives only an `average' energy distribution in an excitation with time-evolving structure, this theory is insufficient for those cases where the non-stationary processes cannot be modeled as stationary or quasi-stationary. As a natural extension to double Fourier Transform for such processes is not considered to be practical, a large amount of effort has been devoted to modeling a (slowly-varying) non-stationary process through modulating function-based power spectral density function (PSDF). The auto power spectral density function of the response as a result of random wind and earthquake excitations with respect to general coordinates has the form

$$S_{q_i q_i}(\Omega) = \sum_{r=1}^{6} \sum_{s=1}^{6} H_r^*(\Omega)\, H_s(\Omega)\, S_{f_r f_s}(\Omega)$$

$$\begin{aligned} S_{q_i q_i}(\Omega) = {} & H_1^*(\Omega)\, H_1(\Omega)\, S_{\xi\xi}(\Omega) + H_1^*(\Omega)\, H_2(\Omega)\, S_{\xi\eta}(\Omega) + \ldots\ldots + H_1^*(\Omega)\, H_6(\Omega)\, S_{\xi f_6}(\Omega) + \\ & H_2^*(\Omega)\, H_1(\Omega)\, S_{\eta\xi}(\Omega) + H_2^*(\Omega)\, H_2(\Omega)\, S_{\eta\eta}(\Omega) + \ldots\ldots + H_2^*(\Omega)\, H_6(\Omega)\, S_{\eta f_6}(\Omega) + \ldots\ldots + \\ & H_6^*(\Omega)\, H_1(\Omega)\, S_{f_6\xi}(\Omega) + H_6^*(\Omega)\, H_2(\Omega)\, S_{f_6\eta}(\Omega) + \ldots\ldots + H_6^*(\Omega)\, H_6(\Omega)\, S_{f_6 f_6}(\Omega) \end{aligned} \tag{35}$$

The cross correlation function of excitation functions with respect to general coordinates is

$$\begin{aligned} R_{Q_i Q_j}(\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} Q_i(t)\, Q_j(t+\tau)\, dt \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [\chi_{1i} f_1(t) + \chi_{2i} f_2(t) + \ldots + \chi_{6i} f_6(t)][\chi_{1j} f_1(t+\tau) + \chi_{2j} f_2(t+\tau) + \ldots + \chi_{6j} f_6(t+\tau)]\, dt \\ &= \chi_{1i}\chi_{1j} R_{f_1 f_1}(\tau) + \chi_{1i}\chi_{2j} R_{f_1 f_2}(\tau) + \ldots + \chi_{1i}\chi_{6j} R_{f_1 f_6}(\tau) + \chi_{2i}\chi_{1j} R_{f_2 f_1}(\tau) + \chi_{2i}\chi_{2j} R_{f_2 f_2}(\tau) \\ &\quad + \ldots + \chi_{2i}\chi_{6j} R_{f_2 f_6}(\tau) + \ldots + \chi_{6i}\chi_{1j} R_{f_6 f_1}(\tau) + \chi_{6i}\chi_{2j} R_{f_6 f_2}(\tau) + \ldots + \chi_{6i}\chi_{6j} R_{f_6 f_6}(\tau) \end{aligned} \tag{36}$$

The cross and auto power spectral density functions of excitation functions are

$$S_{Q_i Q_j}(\Omega) = \sum_{k=1}^{N} \sum_{l=1}^{N} \chi_{ki}\, \chi_{lj}\, S_{f_k f_l}(\Omega) \quad , \quad S_{f_k f_l}(\Omega) = HA_k^*(\Omega).HA_l(\Omega).S_{kl}(\Omega) \quad ,$$

$$S_{f_1 f_1}(\Omega) = HA_1^*(\Omega).HA_1(\Omega).S_{\xi\xi}(\Omega) \tag{37}$$

$$f_1(t) = u(1).\xi(t) + u(2).\dot{\xi}(t) \quad ,$$

$$f_1(\Omega) = u(1).\xi(\Omega) + i\Omega\, u(2).\xi(\Omega) = [u(1) + i\Omega\, u(2)].\xi(\Omega) = [u(1) + i\Omega\, u(2)].[\dot{\xi}(\Omega) / i\Omega]$$

The excitation functions can be represented as

$$Q_i(t) = \sum_{n=1}^{6} \chi_{ni}\, f_n(t) \quad , \quad Q_r(t) = \sum_{i=1}^{6} \chi_{ir}\, f_i(t) \quad , \quad Q_s(t) = \sum_{j=1}^{6} \chi_{js}\, f_j(t) \quad ,$$

$$Q_r(t)\, Q_s(t+\tau) = \sum_{i=1}^{6} \sum_{j=1}^{6} \chi_{ir}\, f_i(t) . \chi_{js}\, f_j(t) \tag{38}$$

$$Q_1(t)=\frac{1}{\omega_1^2}.\underline{\chi}_{(1)}^T\begin{bmatrix} f_1(t)\\ f_2(t)\\ f_3(t)\\ f_4(t)\\ f_5(t)\\ f_6(t)\end{bmatrix}=\frac{1}{\omega_1^2}.\underline{\chi}_{(1)}^T\begin{bmatrix} k_{EV}\xi(t)+r_{EV}\dot{\xi}(t)\\ k_{EH}\eta(t)+r_{EH}\dot{\eta}(t)\\ 0\\ 0\\ \int^A C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA\\ \int^A (z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA\end{bmatrix}=\frac{1}{\omega_1^2}.\underline{\chi}_{(1)}^T\begin{bmatrix} u(1)\xi(t)+u(2)\dot{\xi}(t)\\ u(3)\eta(t)+u(4)\dot{\eta}(t)\\ 0\\ 0\\ u(5)v(t)\\ u(6)w(t)\end{bmatrix}$$

$$Q_2(t)=\frac{1}{\omega_2^2}.\underline{\chi}_{(2)}^T.\underline{f}(t)\quad,\quad Q_3(t)=\frac{1}{\omega_3^2}.\underline{\chi}_{(3)}^T.\underline{f}(t)\quad,\quad Q_4(t)=\frac{1}{\omega_4^2}.\underline{\chi}_{(4)}^T.\underline{f}(t)\quad,\quad Q_5(t)=\frac{1}{\omega_5^2}.\underline{\chi}_{(5)}^T.\underline{f}(t)\quad,$$

$$Q_6(t)=\frac{1}{\omega_6^2}.\underline{\chi}_{(6)}^T.\underline{f}(t) \tag{39}$$

The cross correlation function of excitations is

$$R_{Q_rQ_s}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}Q_r(t)\,Q_s(t+\tau)\,dt=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}[\frac{1}{\omega_r^2}\underline{\chi}_{(r)}^T f(t)].[\frac{1}{\omega_s^2}\underline{\chi}_{(s)}^T f(t+\tau)]\,dt$$

$$=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\underline{\chi}_{ir}\underline{\chi}_{js}f_i(t)f_j(t+\tau)\,dt=\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\underline{\chi}_{ir}\underline{\chi}_{js}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}f_i(t)f_j(t+\tau)\,dt$$

$$=\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\underline{\chi}_{ir}\underline{\chi}_{js}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}f_i(t)f_j(t+\tau)\,dt=\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\underline{\chi}_{ir}\underline{\chi}_{js}.R_{f_if_j}(\tau)$$

$$=\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}.[\chi_{1r}\chi_{1s}.R_{f_1f_1}(\tau)+\chi_{5r}\chi_{5s}.R_{f_5f_5}(\tau)+\chi_{5r}\chi_{6s}.R_{f_5f_6}(\tau)+\chi_{6r}\chi_{5s}.R_{f_6f_5}(\tau)+\chi_{6r}\chi_{6s}.R_{f_6f_6}(\tau)]$$

$$=\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}.\{\chi_{1r}\chi_{1s}[k_{EV}^2.R_{\xi\xi}(\tau)+k_{EV}.r_{EV}.R_{\xi\dot{\xi}}(\tau)+r_{EV}.k_{EV}.R_{\dot{\xi}\xi}(\tau)+r_{EV}^2.R_{\dot{\xi}\dot{\xi}}(\tau)]+$$
$$[\chi_{5r}\chi_{5s}.R_{f_5f_5}(\tau)+\chi_{5r}\chi_{6s}.R_{f_5f_6}(\tau)+\chi_{6r}\chi_{5s}.R_{f_6f_5}(\tau)+\chi_{6r}\chi_{6s}.R_{f_6f_6}(\tau)]\}$$

The cross power spectral density function of excitation functions has the form

$$S_{Q_rQ_s}(\Omega)=\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}.\{\chi_{1r}\chi_{1s}[k_{EV}^2.S_{\xi\xi}(\tau)+k_{EV}.r_{EV}.S_{\xi\dot{\xi}}(\tau)+r_{EV}.k_{EV}.S_{\dot{\xi}\xi}(\tau)+r_{EV}^2.S_{\dot{\xi}\dot{\xi}}(\tau)]+$$

$$C_f^2.\rho^2.\overline{U}^2(z).S_{uu}(\Omega)[\chi_{5r}\chi_{5s}.|X_{11}(\Omega)|^2+\chi_{5r}\chi_{6s}.|X_{12}(\Omega)|^2+\chi_{6r}\chi_{5s}.|X_{21}(\Omega)|^2+\chi_{6r}\chi_{6s}.|X_{22}(\Omega)|^2]\}$$

$$S_{Q_rQ_s}(\Omega)=\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\underline{\chi}_{ir}\underline{\chi}_{js}.S_{f_if_j}(\Omega) \tag{40}$$

The differential equations of motion can be written in the form

$$\ddot{q}_i(t) + \frac{R_i}{m_i}\dot{q}_i(t) + \frac{k_i}{m_i}q_i(t) = \frac{1}{m_i}.Q_i^{'}(t) = \omega_i^2.Q_i(t)$$

$$\ddot{q}_i(t) + 2D_i\omega_i.\dot{q}_i(t) + \omega_i^2.q_i(t) = \frac{1}{k_i}\omega_i^2.Q_i^{'}(t) = \omega_i^2\frac{1}{k_i}.Q_i^{'}(t) = \omega_i^2.Q_i(t) \quad \text{with} \quad Q_i(t) = \frac{1}{\omega_i^2}\frac{1}{m_i}.Q_i^{'}(t) \tag{41}$$

The cross power spectral density function of the vibration response with respect to general coordinates is

$$S_{q_r q_s}(\Omega) = H_r^*(\Omega).H_s(\Omega).S_{Q_r Q_s}(\Omega) \tag{42}$$

The cross power spectral density function of the vibration response with respect to original coordinates is

$$S_{X_r X_s}(\Omega) = \sum_{i=1}^{n}\sum_{j=1}^{n}\chi_{ri}\chi_{sj}.S_{q_i q_j}(\Omega) \tag{43}$$

Substitute from Eq. 42 in Eq. 46 results in

$$S_{X_r X_s}(\Omega) = \sum_{i=1}^{n}\sum_{j=1}^{n}\chi_{ri}\chi_{sj}.H_i^*(\Omega).H_j(\Omega).S_{Q_i Q_j}(\Omega) \tag{44}$$

Substitute from Eq. 40 in Eq. 44, one can obtain the cross power spectral density function of the response with respect to original coordinates of the form

$$S_{X_r X_s}(\Omega) = \sum_{i=1}^{n}\sum_{j=1}^{n}\chi_{ri}\chi_{sj}.H_i^*(\Omega).H_j(\Omega).(\frac{1}{m_i}.\frac{1}{\omega_i^2}).(\frac{1}{m_j}.\frac{1}{\omega_j^2}).\sum_{k=1}^{n}\sum_{l=1}^{n}\chi_{ki}\chi_{lj}.S_{f_k f_l}(\Omega) \tag{45}$$

and the auto power spectral density function of the response with respect to original coordinates of the form

$$S_{X_n X_n}(\Omega) = \sum_{i=1}^{6}\sum_{j=1}^{6}\chi_{ni}\chi_{nj}.H_i^*(\Omega).H_j(\Omega).\frac{1}{k_i}.\frac{1}{k_j}\sum_{r=1}^{6}\sum_{s=1}^{6}\chi_{ri}\chi_{sj}.S_{f_r f_s}(\Omega) \tag{46}$$

4.4 The power spectral density function of the excitations

4.4.1 Correlation function of the excitations

The correlation function of the excitations with respect to general coordinates is

$$R_{Q_r Q_s}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}Q_r(t)\,Q_s(t+\tau)\,dt \tag{47}$$

Substitute from Eq. 32 in Eq. 47 results in

$$R_{Q_r Q_s}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{1}{\omega_r^2}[B_{r1}\xi(t) + B_{r2}\dot{\xi}(t) + B_{r3}\eta(t) + B_{r4}\dot{\eta}(t) + B_{r5}\int^{A} C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA +$$

$$B_{r6}\int^{A}(z-\frac{c}{2}).\,C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\frac{1}{\omega_s^2}[B_{s1}\xi(t+\tau)+B_{s2}\dot{\xi}(t+\tau)+B_{s3}\eta(t+\tau)+$$

$$B_{s4}\dot{\eta}(t+\tau)+B_{s5}\int^{A}C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+B_{s6}\int^{A}(z-\frac{c}{2}).\,C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA]\,dt$$

$$R_{Q_rQ_s}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\{\ B_{r1}B_{s1}\xi(t)\xi(t+\tau)+B_{r1}B_{s2}\xi(t)\dot{\xi}(t+\tau)+B_{r1}B_{s3}\xi(t)\eta(t+\tau)+B_{r1}B_{s4}\xi(t)\dot{\eta}(t+\tau)+$$

$$B_{r1}B_{s5}\xi(t)\int^{A}C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+B_{r1}B_{s6}\xi(t)\int^{A}(z-\frac{c}{2}).\,C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+$$

$$B_{r2}B_{s1}\dot{\xi}(t)\xi(t+\tau)+B_{r2}B_{s2}\dot{\xi}(t)\dot{\xi}(t+\tau)+B_{r2}B_{s3}\dot{\xi}(t)\eta(t+\tau)+B_{r2}B_{s4}\dot{\xi}(t)\dot{\eta}(t+\tau)+$$

$$B_{r2}B_{s5}\dot{\xi}(t)\int^{A}C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+B_{r2}B_{s6}\dot{\xi}(t)\int^{A}(z-\frac{c}{2}).\,C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+$$

$$B_{r3}B_{s1}\eta(t)\xi(t+\tau)+B_{r3}B_{s2}\eta(t)\dot{\xi}(t+\tau)+B_{r3}B_{s3}\eta(t)\eta(t+\tau)+B_{r3}B_{s4}\eta(t)\dot{\eta}(t+\tau)+$$

$$B_{r3}B_{s5}\eta(t)\int^{A}C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+B_{r3}B_{s6}\eta(t)\int^{A}(z-\frac{c}{2}).\,C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+$$

$$B_{r4}B_{s1}\dot{\eta}(t)\xi(t+\tau)+B_{r4}B_{s2}\dot{\eta}(t)\dot{\xi}(t+\tau)+B_{r4}B_{s3}\dot{\eta}(t)\eta(t+\tau)+B_{r4}B_{s4}\dot{\eta}(t)\dot{\eta}(t+\tau)+$$

$$B_{r4}B_{s5}\dot{\eta}(t)\int^{A}C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+B_{r4}B_{s6}\dot{\eta}(t)\int^{A}(z-\frac{c}{2}).\,C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+$$

$$B_{r5}B_{s1}[\int^{A}C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\xi(t+\tau)+B_{r5}B_{s2}[\int^{A}C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\dot{\xi}(t+\tau)+$$

$$B_{r5}B_{s3}[\int^{A}C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\eta(t+\tau)+B_{r5}B_{s4}[\int^{A}C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\dot{\eta}(t+\tau)+$$

$$B_{r5}B_{s5}[\int^{A}C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].[\int^{A}C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA]+$$

$$B_{r5}B_{s6}[\int^{A}C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].[\int^{A}(z-\frac{c}{2}).\,C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA]+$$

$$B_{r6}B_{s1}[\int^{A}(z-\frac{c}{2}).\,C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\xi(t+\tau)+B_{r6}B_{s2}[\int^{A}(z-\frac{c}{2}).\,C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\dot{\xi}(t+\tau)+$$

$$B_{r6}B_{s3}[\int^{A}(z-\frac{c}{2}).\,C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\eta(t+\tau)+B_{r6}B_{s4}[\int^{A}(z-\frac{c}{2}).\,C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\dot{\eta}(t+\tau)+$$

$$B_{r6}B_{s5}[\int^{A}(z-\frac{c}{2}).\,C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].[\int^{A}C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA]+$$

$$B_{r6}B_{s6}[\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].[\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA]\ \}dt$$

$$R_{Q_rQ_s}(\tau)=\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\ \{\ B_{r1}B_{s1}R_{\xi\xi}(\tau)+B_{r1}B_{s2}R_{\xi\dot{\xi}}(\tau)+B_{r1}B_{s3}R_{\xi\eta}(\tau)+B_{r1}B_{s4}R_{\xi\dot{\eta}}(\tau)+$$

$$B_{r1}B_{s5}\int^{A}C_p.\rho.\overline{U}(z).R_{\xi U^{'}}(\tau)\,dA+B_{r1}B_{s6}\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).R_{\xi U^{'}}(\tau)\,dA+$$

$$B_{r2}B_{s1}R_{\dot{\xi}\xi}(\tau)+B_{r2}B_{s2}R_{\dot{\xi}\dot{\xi}}(\tau)+B_{r2}B_{s3}R_{\dot{\xi}\eta}(\tau)+B_{r2}B_{s4}R_{\dot{\xi}\dot{\eta}}(\tau)+$$

$$B_{r2}B_{s5}\int^{A}C_p.\rho.\overline{U}(z).R_{\dot{\xi}U^{'}}(\tau)\,dA+B_{r2}B_{s6}\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).R_{\dot{\xi}U^{'}}(\tau)\,dA+$$

$$B_{r3}B_{s1}.R_{\eta\xi}(\tau)+B_{r3}B_{s2}.R_{\eta\dot{\xi}}(\tau)+B_{r3}B_{s3}R_{\eta\eta}(\tau)+B_{r3}B_{s4}.R_{\eta\dot{\eta}}(\tau)+$$

$$B_{r3}B_{s5}\int^{A}C_p.\rho.\overline{U}(z).R_{\eta U^{'}}(\tau)dA+B_{r3}B_{s6}\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).R_{\eta U^{'}}(\tau)\,dA+$$

$$B_{r4}B_{s1}R_{\dot{\eta}\xi}(\tau)+B_{r4}B_{s2}R_{\dot{\eta}\dot{\xi}}(\tau)+B_{r4}B_{s3}R_{\dot{\eta}\eta}(\tau)+B_{r4}B_{s4}R_{\dot{\eta}\dot{\eta}}(\tau)+$$

$$B_{r4}B_{s5}\int^{A}C_p.\rho.\overline{U}(z).R_{\dot{\eta}U^{'}}(\tau)\,dA+B_{r4}B_{s6}\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).R_{\dot{\eta}U^{'}}(\tau)\,dA+$$

$$B_{r5}B_{s1}\int^{A}C_p.\rho.\overline{U}(z).R_{U^{'}\xi}(\tau)\,dA+B_{r5}B_{s2}\int^{A}C_p.\rho.\overline{U}(z).R_{U^{'}\dot{\xi}}(\tau)\,dA+B_{r5}B_{s3}\int^{A}C_p.\rho.\overline{U}(z).R_{U^{'}\eta}(\tau)\,dA+$$

$$B_{r5}B_{s4}\int^{A}C_p.\rho.\overline{U}(z).R_{U^{'}\dot{\eta}}(\tau)\,dA+B_{r5}B_{s5}\int^{A}\int^{A}C_p^2.\rho^2.\overline{U}(z_1).\overline{U}(z_2).R_{U_1^{'}U_2^{'}}(\tau).dA_1.dA_2$$

$$B_{r5}B_{s6}\int^{A_1}\int^{A_2}C_p^2.\rho^2.\overline{U}(z_1).\overline{U}(z_2).(z_2-\frac{c}{2})R_{U_1^{'}U_2^{'}}(\tau).dA_1.dA_2+B_{r6}B_{s1}\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).R_{U^{'}\xi}(\tau)\,dA+$$

$$B_{r6}B_{s2}\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).R_{U^{'}\dot{\xi}}(\tau)\,dA+B_{r6}B_{s3}\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).R_{U^{'}\eta}(\tau)\,dA+$$

$$B_{r6}B_{s4}[\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).R_{U^{'}\dot{\eta}}(\tau)\,dA+B_{r6}B_{s5}\int^{A_1}\int^{A_2}C_p^2.\rho^2.\overline{U}(z_1).\overline{U}(z_2).(z_1-\frac{c}{2}).R_{U_2^{'}U_1^{'}}(\tau).dA_1.dA_2+$$

$$B_{r6}B_{s6}\int^{A_1}\int^{A_2}(z_1-\frac{c}{2})(z_2-\frac{c}{2})C_p^2.\rho^2.\overline{U}(z_1).\overline{U}(z_2).R_{U_1^{'}U_2^{'}}(\tau).dA_1.dA_2\ \} \qquad (48)$$

Since the wind velocity U(z,t) and the underground excitations $\xi(t),\eta(t)$ are uncorrelated, the following correlation functions must have the values of zero.

$$R_{\xi U^{'}}(\tau)=R_{\dot{\xi}U^{'}}(\tau)=R_{\eta U^{'}}(\tau)=R_{\dot{\eta}U^{'}}(\tau)=0 \text{ and } R_{U^{'}\xi}(\tau)=R_{U^{'}\dot{\xi}}(\tau)=R_{U^{'}\eta}(\tau)=R_{U^{'}\dot{\eta}}(\tau)=0 \qquad (49)$$

4.4.2 The power spectral density function of the excitations

The cross power spectral density function of the excitations with respect to general coordinates is

$$S_{Q_rQ_s}(\Omega) = F\{R_{Q_rQ_s}(\tau)\} = \int_{-\infty}^{\infty} R_{Q_rQ_s}(\tau)\, e^{-i\Omega\tau} d\tau \tag{50}$$

Substitute from Eqs. 48 and 49 in Eq. 50 results in

$$\begin{aligned}
S_{Q_rQ_s}(\Omega) = \int_{-\infty}^{\infty} \frac{1}{\omega_r^2}\frac{1}{\omega_s^2} \Big\{ & B_{r1}B_{s1}R_{\xi\xi}(\tau) + B_{r1}B_{s2}R_{\xi\dot{\xi}}(\tau) + B_{r1}B_{s3}R_{\xi\eta}(\tau) + B_{r1}B_{s4}R_{\xi\dot{\eta}}(\tau) + B_{r2}B_{s1}R_{\dot{\xi}\xi}(\tau) + B_{r2}B_{s2}R_{\dot{\xi}\dot{\xi}}(\tau) + \\
& B_{r2}B_{s3}R_{\dot{\xi}\eta}(\tau) + B_{r2}B_{s4}R_{\dot{\xi}\dot{\eta}}(\tau) + B_{r3}B_{s1}.R_{\eta\xi}(\tau) + B_{r3}B_{s2}.R_{\eta\dot{\xi}}(\tau) + B_{r3}B_{s3}R_{\eta\eta}(\tau) + B_{r3}B_{s4}.R_{\eta\dot{\eta}}(\tau) + \\
& B_{r4}B_{s1}R_{\dot{\eta}\xi}(\tau) + B_{r4}B_{s2}R_{\dot{\eta}\dot{\xi}}(\tau) + B_{r4}B_{s3}R_{\dot{\eta}\eta}(\tau) + B_{r4}B_{s4}R_{\dot{\eta}\dot{\eta}}(\tau) + \\
& B_{r5}B_{s5}\int^{A}\int^{A} C_p^2.\rho^2.\overline{U}(z_1)\overline{U}(z_2).R_{U_1'U_2'}(\tau).dA_1.dA_2 + \\
& B_{r5}B_{s6}\int^{A_1}\int^{A_2} C_p^2.\rho^2.\overline{U}(z_1)\overline{U}(z_2).(z_2-\frac{c}{2})R_{U_1'U_2'}(\tau).dA_1.dA_2 + \\
& B_{r6}B_{s5}\int^{A_1}\int^{A_2} (z_1-\frac{c}{2})\, C_p^2.\rho^2.\overline{U}(z_1).\overline{U}(z_2).R_{U_1'U_2'}(\tau).dA_1.dA_2 + \\
& B_{r6}B_{s6}\int^{A_1}\int^{A_2} (z_1-\frac{c}{2})(z_2-\frac{c}{2})C_p^2.\rho^2.\overline{U}(z_1).\overline{U}(z_2).R_{U_1'U_2'}(\tau).dA_1.dA_2 \Big\} e^{-i\Omega\tau} d\tau
\end{aligned}$$

$$\begin{aligned}
S_{Q_rQ_s}(\Omega) = \frac{1}{\omega_r^2}\frac{1}{\omega_s^2} \Big\{ & B_{r1}B_{s1}S_{\xi\xi}(\Omega) + B_{r1}B_{s2}S_{\xi\dot{\xi}}(\Omega) + B_{r1}B_{s3}S_{\xi\eta}(\Omega) + B_{r1}B_{s4}S_{\xi\dot{\eta}}(\Omega) + B_{r2}B_{s1}S_{\dot{\xi}\xi}(\Omega) + B_{r2}B_{s2}S_{\dot{\xi}\dot{\xi}}(\Omega) + \\
& B_{r2}B_{s3}S_{\dot{\xi}\eta}(\Omega) + B_{r2}B_{s4}S_{\dot{\xi}\dot{\eta}}(\Omega) + B_{r3}B_{s1}.S_{\eta\xi}(\Omega) + B_{r3}B_{s2}.S_{\eta\dot{\xi}}(\Omega) + B_{r3}B_{s3}S_{\eta\eta}(\Omega) + B_{r3}B_{s4}.S_{\eta\dot{\eta}}(\Omega) + \\
& B_{r4}B_{s1}S_{\dot{\eta}\xi}(\Omega) + B_{r4}B_{s2}S_{\dot{\eta}\dot{\xi}}(\Omega) + B_{r4}B_{s3}S_{\dot{\eta}\eta}(\Omega) + B_{r4}B_{s4}S_{\dot{\eta}\dot{\eta}}(\Omega) + \\
& B_{r5}B_{s5}\int^{A}\int^{A} C_p^2.\rho^2.\overline{U}(z_1)\overline{U}(z_2)S_{U_1U_2}(\Omega).dA_1.dA_2 + \\
& B_{r5}B_{s6}\int^{A_1}\int^{A_2} C_p^2.\rho^2.\overline{U}(z_1)\overline{U}(z_2).(z_2-\frac{c}{2})S_{U_1U_2}(\Omega).dA_1.dA_2 + \\
& B_{r6}B_{s5}\int^{A_1}\int^{A_2} (z_1-\frac{c}{2})\, C_p^2.\rho^2.\overline{U}(z_1).\overline{U}(z_2).S_{U_1U_2}(\Omega).dA_1.dA_2 + \\
& B_{r6}B_{s6}\int^{A_1}\int^{A_2} (z_1-\frac{c}{2})(z_2-\frac{c}{2})C_p^2.\rho^2.\overline{U}(z_1).\overline{U}(z_2).S_{U_1U_2}(\Omega).dA_1.dA_2 \Big\}
\end{aligned} \tag{51}$$

The wind velocity $\overline{U}(z)$ depends on the height of the building, according to the following equation

$$\overline{U}(z) = (\frac{z}{H})^{\alpha}\, \overline{U}(H) \tag{52}$$

Using Eq. 52 in Eq. 51

$$\begin{aligned}
S_{Q_rQ_s}(\Omega) = \frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\ \{\ & B_{r1}B_{s1}S_{\xi\xi}(\Omega) + B_{r1}B_{s2}S_{\xi\dot{\xi}}(\Omega) + B_{r1}B_{s3}S_{\xi\eta}(\Omega) + B_{r1}B_{s4}S_{\xi\dot{\eta}}(\Omega) + B_{r2}B_{s1}S_{\dot{\xi}\xi}(\Omega) + \\
& B_{r2}B_{s2}S_{\dot{\xi}\dot{\xi}}(\Omega) + B_{r2}B_{s3}S_{\dot{\xi}\eta}(\Omega) + B_{r2}B_{s4}S_{\dot{\xi}\dot{\eta}}(\Omega) + B_{r3}B_{s1}.S_{\eta\xi}(\Omega) + B_{r3}B_{s2}.S_{\eta\dot{\xi}}(\Omega) + B_{r3}B_{s3}S_{\eta\eta}(\Omega) + \\
& B_{r3}B_{s4}.S_{\eta\dot{\eta}}(\Omega) + B_{r4}B_{s1}S_{\dot{\eta}\xi}(\Omega) + B_{r4}B_{s2}S_{\dot{\eta}\dot{\xi}}(\Omega) + B_{r4}B_{s3}S_{\dot{\eta}\eta}(\Omega) + B_{r4}B_{s4}S_{\dot{\eta}\dot{\eta}}(\Omega) + \\
& B_{r6}B_{s5}.C_f^2.\rho^2.\overline{U}^2(H).S_U(\Omega)\int^{A_1}\int^{A_2}(\frac{z_1}{H})^{\alpha}(z_1-\frac{c}{2})(\frac{z_2}{H})^{\alpha}\gamma_{U_1U_2}(\Omega).dA_1.dA_2 + \\
& B_{r5}B_{s6}.C_f^2.\rho^2.\overline{U}^2(H).S_U(\Omega)\int^{A_1}\int^{A_2}(\frac{z_1}{H})^{\alpha}(\frac{z_2}{H})^{\alpha}(z_2-\frac{c}{2})\gamma_{U_1U_2}(\Omega).dA_1.dA_2 + \\
& B_{r6}B_{s6}.C_f^2.\rho^2.\overline{U}^2(H).S_U(\Omega)\int^{A_1}\int^{A_2}(\frac{z_1}{H})^{\alpha}(z_1-\frac{c}{2})(\frac{z_2}{H})^{\alpha}(z_2-\frac{c}{2})\gamma_{U_1U_2}(\Omega).dA_1.dA_2
\end{aligned} \tag{53}$$

These double integrals can be described as Aerodynamic Amplification Functions (Transformation Functions) are

$$\begin{aligned}
|X_{11}(\Omega)|^2 &= \int^{A_1}\int^{A_2}(\frac{z_1}{H})^{\alpha}(\frac{z_2}{H})^{\alpha}\gamma_{U_1U_2}(\Omega).dA_1.dA_2 \quad , \\
|X_{12}(\Omega)|^2 &= \int^{A_1}\int^{A_2}(\frac{z_1}{H})^{\alpha}(\frac{z_2}{H})^{\alpha}(z_2-\frac{c}{2})\gamma_{U_1U_2}(\Omega).dA_1.dA_2 \\
|X_{21}(\Omega)|^2 &= \int^{A_1}\int^{A_2}(\frac{z_1}{H})^{\alpha}(z_1-\frac{c}{2})(\frac{z_2}{H})^{\alpha}\gamma_{U_1U_2}(\Omega).dA_1.dA_2 \ , \\
&\qquad\qquad \} \\
|X_{22}(\Omega)|^2 &= \int^{A_1}\int^{A_2}(\frac{z_1}{H})^{\alpha}(z_1-\frac{c}{2})(\frac{z_2}{H})^{\alpha}(z_2-\frac{c}{2})\gamma_{U_1U_2}(\Omega).dA_1.dA_2
\end{aligned} \tag{54}$$

$$\begin{aligned}
S_{Q_rQ_s}(\Omega) = \frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\ \{\ & B_{r1}B_{s1}S_{\xi\xi}(\Omega) + B_{r1}B_{s2}S_{\xi\dot{\xi}}(\Omega) + B_{r1}B_{s3}S_{\xi\eta}(\Omega) + B_{r1}B_{s4}S_{\xi\dot{\eta}}(\Omega) + \\
& B_{r2}B_{s1}S_{\dot{\xi}\xi}(\Omega) + B_{r2}B_{s2}S_{\dot{\xi}\dot{\xi}}(\Omega) + B_{r2}B_{s3}S_{\dot{\xi}\eta}(\Omega) + B_{r2}B_{s4}S_{\dot{\xi}\dot{\eta}}(\Omega) + B_{r3}B_{s1}.S_{\eta\xi}(\Omega) + B_{r3}B_{s2}.S_{\eta\dot{\xi}}(\Omega) + \\
& B_{r3}B_{s3}S_{\eta\eta}(\Omega) + B_{r3}B_{s4}.S_{\eta\dot{\eta}}(\Omega) + B_{r4}B_{s1}S_{\dot{\eta}\xi}(\Omega) + B_{r4}B_{s2}S_{\dot{\eta}\dot{\xi}}(\Omega) + B_{r4}B_{s3}S_{\dot{\eta}\eta}(\Omega) + B_{r4}B_{s4}S_{\dot{\eta}\dot{\eta}}(\Omega) + \\
& C_f^2.\rho^2.\overline{U}^2(H).S_U(\Omega)\,[B_{r5}B_{s5}.|X_{11}(\Omega)|^2 + B_{r5}B_{s6}.|X_{12}(\Omega)|^2 + B_{r6}B_{s5}.|X_{21}(\Omega)|^2 + B_{r6}B_{s6}.|X_{22}(\Omega)|^2]
\end{aligned}$$

Auto power spectral density function of the excitation with respect to the first general coordinates

$$
\begin{aligned}
S_{Q_1Q_1}(\Omega) = \frac{1}{\omega_1^2}\frac{1}{\omega_1^2}\, \{\, & B_{11}B_{11}S_{\xi\xi}(\Omega) + B_{11}B_{12}S_{\xi\dot{\xi}}(\Omega) + B_{11}B_{13}S_{\xi\eta}(\Omega) + B_{11}B_{14}S_{\xi\dot{\eta}}(\Omega) + B_{12}B_{11}S_{\dot{\xi}\xi}(\Omega) + B_{12}B_{12}S_{\dot{\xi}\dot{\xi}}(\Omega) + \\
& B_{12}B_{13}S_{\dot{\xi}\eta}(\Omega) + B_{12}B_{14}S_{\dot{\xi}\dot{\eta}}(\Omega) + B_{13}B_{11}.S_{\eta\xi}(\Omega) + B_{13}B_{12}.S_{\eta\dot{\xi}}(\Omega) + B_{13}B_{13}S_{\eta\eta}(\Omega) + \\
& B_{13}B_{14}.S_{\eta\dot{\eta}}(\Omega) + B_{14}B_{11}S_{\dot{\eta}\xi}(\Omega) + B_{14}B_{12}S_{\dot{\eta}\dot{\xi}}(\Omega) + B_{14}B_{13}S_{\dot{\eta}\eta}(\Omega) + B_{14}B_{14}S_{\dot{\eta}\dot{\eta}}(\Omega) + \\
& C_f^2.\rho^2.\bar{U}^2(H).S_U(\Omega)\,[B_{15}B_{15}.|X_{11}(\Omega)|^2 + B_{15}B_{16}.|X_{12}(\Omega)|^2 + B_{16}B_{15}.|X_{21}(\Omega)|^2 + B_{16}B_{16}.|X_{22}(\Omega)|^2]
\end{aligned}
\tag{55}
$$

4.5 Complex transformation matrix with respect to general coordinates

Fourier transformation of the vibration response and excitation has the following form

$$q_n(\Omega).[-\Omega^2 + i\, 2D_n\omega_n\Omega + \omega_n^2] = \omega_n^2.\, Q_n(\Omega)$$

Where $q_n(\Omega) = H_n(\Omega).\, Q_n(\Omega)$ with $H_n(\Omega) = \dfrac{1}{[1 - (\frac{\Omega}{\omega_n})^2] + i\,[2D_n(\frac{\Omega}{\omega_n})]}$, $n = 1, 2, ..., 6$

(56)

and its absolute value is $\quad AHF(\Omega) = \dfrac{1}{[1 - (\frac{\Omega}{\omega_n})^2]^2 + [2\,D_n(\frac{\Omega}{\omega_n})]^2}$, $n = 1, 2, ..., 6$

4.6 Response power spectral density function with respect to general coordinates

Cross correlation functions of the response with respect to general coordinates have the form

$$R_{q_rq_s}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} q_r(t)\, q_s(t+\tau)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r^*(\Omega)\, H_s(\Omega)\, S_{Q_rQ_s}(\Omega)\, e^{i\Omega\tau}\, d\Omega \tag{57}$$

The response power spectral density function with respect to general coordinates is

Cross: $S_{q_rq_s}(\Omega) = F\{R_{q_rq_s}(\tau)\}$ and Auto: $S_{q_n}(\Omega) = |H_n(\Omega)|^2.\, S_{Q_n}(\Omega)$

Auto power spectral density function for n-eigen form with respect to general coordinates is

$$
\begin{aligned}
S_{q_nq_n}(\Omega) = |H_n(\Omega)|^2.\frac{1}{\omega_n^2}\, \{\, & B_{n1}B_{n1}S_{\xi\xi}(\Omega) + B_{n1}B_{n2}S_{\xi\dot{\xi}}(\Omega) + B_{n1}B_{n3}S_{\xi\eta}(\Omega) + B_{n1}B_{n4}S_{\xi\dot{\eta}}(\Omega) + B_{n2}B_{n1}S_{\dot{\xi}\xi}(\Omega) + \\
& B_{n2}B_{n2}S_{\dot{\xi}\dot{\xi}}(\Omega) + B_{n2}B_{n3}S_{\dot{\xi}\eta}(\Omega) + B_{n2}B_{n4}S_{\dot{\xi}\dot{\eta}}(\Omega) + B_{n3}B_{n1}.S_{\eta\xi}(\Omega) + B_{n3}B_{n2}.S_{\eta\dot{\xi}}(\Omega) + \\
& B_{n3}B_{n3}S_{\eta\eta}(\Omega) + B_{n3}B_{n4}.S_{\eta\dot{\eta}}(\Omega) + B_{n4}B_{n1}S_{\dot{\eta}\xi}(\Omega) + B_{n4}B_{n2}S_{\dot{\eta}\dot{\xi}}(\Omega) + B_{n4}B_{n3}S_{\dot{\eta}\eta}(\Omega) + B_{n4}B_{n4}S_{\dot{\eta}\dot{\eta}}(\Omega) + \\
& C_f^2.\rho^2.\bar{U}^2(H).S_U(\Omega)\,[B_{n5}B_{n5}.|X_{11}(\Omega)|^2 + B_{n5}B_{n6}.|X_{12}(\Omega)|^2 + B_{n6}B_{n5}.|X_{21}(\Omega)|^2 + B_{n6}B_{n6}.|X_{22}(\Omega)|^2]\, \}
\end{aligned}
\tag{58}
$$

Where the mechanical amplification functions (Transformation Functions) are

$$\left|H_n(\Omega)\right|^2 = \frac{1}{[1-(\frac{\Omega}{\omega_n})^2]^2 + i\,[2D_n(\frac{\Omega}{\omega_n})]^2} \quad , \quad n = 1, 2, ..., 6 \tag{59}$$

and the Aerodynamic Amplification Functions (Transformation Functions) are shown in Eqs. 54

4.7 Response power spectral density function with respect to original coordinates

Cross correlation functions of the response with respect to original coordinates have the form

$$R_{X_r X_s}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} X_r(t)\,X_s(t+\tau)\,dt \;= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\sum_{i=1}^{6}\sum_{j=1}^{6}\chi_{ri}\chi_{sj}q_i(t)\,q_j(t+\tau)\,dt$$

$$= \sum_{i=1}^{6}\sum_{j=1}^{6}\chi_{ri}\chi_{sj}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} q_i(t)\,q_j(t+\tau)\,dt = \sum_{i=1}^{6}\sum_{j=1}^{6}\chi_{ri}\chi_{sj}R_{q_i q_j}(\tau) \tag{60}$$

The response power spectral density function with respect to original coordinates is

Cross : $S_{X_r X_s}(\Omega) = F\{R_{X_r X_s}(\tau)\} = \sum_{i=1}^{6}\sum_{j=1}^{6}\chi_{ri}\chi_{sj}S_{q_i q_j}(\Omega)$ and Auto:

$$S_{X_n}(\Omega) = \sum_{i=1}^{6}\sum_{j=1}^{6}\chi_{ni}\chi_{nj}.S_{q_n}(\Omega) \tag{61}$$

$$\begin{aligned}
S_{X_n X_n}(\Omega) = \sum_{i=1}^{6}\sum_{j=1}^{6}\chi_{ni}\chi_{nj}.\left|H_n(\Omega)\right|^2.\frac{1}{\omega_n^4}\{\; & B_{n1}B_{n1}S_{\xi\xi}(\Omega) + B_{n1}B_{n2}S_{\xi\dot{\xi}}(\Omega) + B_{n1}B_{n3}S_{\xi\eta}(\Omega) + B_{n1}B_{n4}S_{\xi\dot{\eta}}(\Omega) + \\
& B_{n2}B_{n1}S_{\dot{\xi}\xi}(\Omega) + B_{n2}B_{n2}S_{\dot{\xi}\dot{\xi}}(\Omega) + B_{n2}B_{n3}S_{\dot{\xi}\eta}(\Omega) + B_{n2}B_{n4}S_{\dot{\xi}\dot{\eta}}(\Omega) + B_{n3}B_{n1}.S_{\eta\xi}(\Omega) + B_{n3}B_{n2}.S_{\eta\dot{\xi}}(\Omega) + \\
& B_{n3}B_{n3}S_{\eta\eta}(\Omega) + B_{n3}B_{n4}.S_{\eta\dot{\eta}}(\Omega) + B_{n4}B_{n1}S_{\dot{\eta}\xi}(\Omega) + B_{n4}B_{n2}S_{\dot{\eta}\dot{\xi}}(\Omega) + B_{n4}B_{n3}S_{\dot{\eta}\eta}(\Omega) + B_{n4}B_{n4}S_{\dot{\eta}\dot{\eta}}(\Omega) + \\
& C_f^2.\rho^2.\overline{U}^2(H).S_U(\Omega)\,[B_{n5}B_{n5}.\left|X_{11}(\Omega)\right|^2 + B_{n5}B_{n6}.\left|X_{12}(\Omega)\right|^2 + B_{n6}B_{n5}.\left|X_{21}(\Omega)\right|^2 + B_{n6}B_{n6}.\left|X_{22}(\Omega)\right|^2]\;\}
\end{aligned} \tag{62}$$

4.8 Mean square value response with respect to original coordinates

Mean square value of the random vibration response with respect to original coordinates can be written as

$$\psi_{X_n}^2 = R_{X_n X_n}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{X_n}(\Omega)\,d\Omega \tag{63}$$

5 Conclusions

This paper outlines a mathematical model describing the vibrations of high-tower buildings and its foundations with general-type equivalent passive springs and dampers, rigid bodies, and some ideal constraints under the effect of randomly fluctuating wind loads and the excitation of earthquake ground motions. Two derivation methods of the equivalent system's differential equations have been considered, namely D'alembert's principle and Lagrange's method, which verified the acceptability of the developed equations of motion. Following conclusions can be withdrawn:

- The mathematical model with 6 degrees of freedom presented in the present paper can be used to investigate the effect of both wind and earthquakes loading.
- Analytical solution of the free vibrations of tall building and its foundation using the general modal analysis method has been performed.
- Analytical solution of forced vibrations of tall building and its foundation has been developed, through the correlation function (time domain) and the power spectral density function (frequency domain) of system response with respect to general and also original coordinates.
- Without wind and earthquakes, structures – particularly large ones – would probably be a lot easier to design and cheaper.
- Random vibrations of building's foundation subjected to seismic excitations of earthquake ground motions and also randomly fluctuating wind pressure fields acting on a building surface are analyzed.

References

Ahearn, E.B. (2010). Reduction of wind-induced vibrations in high-mast light poles. M.S., University of Wyoming, 167 pages; AAT 1491655.

Antonyuk, E.Ya., Timokhin, V.V. (2007). Dynamic response of objects with shock-absorbers to seismic loads. International Applied Mechanics, Vol. 43, No. 12.

Auersch, L. (2008). Dynamic stiffness of foundations on inhomogeneous soils for a realistic prediction of vertical building resonance. J. of Geotechnical and Geoenvironmental Engineering, ASCE.

Balendra, T. et al. (1991). Dynamic response of buildings due to trains in underground tunnels. Earthquake Eng. Struct. Dyn., 20(3), pp. 275–291.

Belakroum, R., et al. (2008). Numerical simulation of dynamic wind loads on rectangular tall buildings. Inter. Review of Mechanical Engineering (I. RE. M.E), Vol. 2, N. 6.

Clouteau, D. et al. (2005). Freefield vibrations due to dynamic loading on a tunnel embedded in a stratified medium. J. Sound Vib., 283(1–2), pp. 173–199.

Davoodi, M., et al. (2008). Modal identification of 1:2 scaled model of a 4-story steel structure by impulse excitation. Seismic Eng. Conf. Commemorating the 1908 Messina and Reggio Calabria Earthquake.

Degrande, G., et al. (2006a). Vibration due to a test train at variable speeds in a deep bored tunnel embedded in London clay. J. Sound Vib., 293(3–5), pp. 626–644.

Degrande, G., et al. (2006b). A numerical model for ground-borne vibrations from underground railway traffic based on a periodic finite element—Boundary element formulation. J. Sound Vib., 293(3–5), pp. 645–666.

Forrest, J. A., and Hunt, H. E. M. (2006a). Ground vibration generated by trains in underground tunnels. J. Sound Vib., 294(4–5), pp. 706–736.

Forrest, J. A., and Hunt, H. E. M. (2006b). A three-dimensional tunnel model for calculation of train-induced round vibration. J. Sound Vib., 294 (4–5), pp. 678–705.

Ghafari Oskoei, S.A. (2011). Earthquake-resistant design procedures for tall guyed telecommunication masts. Ph.D., McGill University (Canada), 235 pages; AAT NR74516.

Gong, Y. (2010). A New free vibration analysis method for space mega frames of super tall buildings. Materials Science and Engineering 10**,** 012002 doi:10.1088/1757-899X/10/1/012002.

Gouasmia, A. et al. (2009). Direct approach to seismic soil-structure-interaction analysis: Building group case. Inter. Review of Mechanical Engineering (I.RE.M.E.), vol. 3, M. 5.

Gupta, S., et al. (2007). A comparison of two numerical models for the prediction of vibrations from underground railway traffic. Soil Dyn. Earthquake Eng., 27(7), pp. 608–624.

Kim, S.J. (2008). Seismic assessment of RC structures considering vertical ground motion. University of Illinois at Urbana-Champaign, ISBN: 9781109026320.

Kliukas, R. et al. (2008). Investigation of traffic-induced vibration in Vilnius Arch-Cathedral Belfry. Transport 23(4), pp. 323–329.

Krier, D. (2009). Modeling of integral abutment bridges considering soil-structure interaction effects. The University of Oklahoma, ISBN: 9781109451252.

Kurzweil, L. G. (1979). Ground-borne noise and vibration from underground rail systems. J. Sound Vib., 66(3), pp. 363–370.

Kuźniar, K. and Waszczyszyn, Z. (2006). Neural networks and principal component analysis for identification of building natural periods. J. of Computing in Civil Engineering 20, ASCE, pp.431-436.

Lorenz, H. (1955). Dynamik im Grundbau. Grundbau Taschenbuch, B.I. Berlin. pp. 205.

Melke, J. (1988). Noise and vibration from underground railway lines : proposals for a prediction procedure. J. Sound Vib., 120(2), 391–406.

Metrikine, A. V., and Vrouwenvelder, A.C.W.M. (2000). Surface ground vibration due to moving train in a tunnel: Two-dimensional model. J. Sound Vib., 234(1), pp. 43–66.

Panagiotou, M. (2008). Seismic design, testing and analysis of reinforced concrete wall buildings. University of California, San Diego, ISBN: 9780549561705.

Prowell, I. (2011). An Experimental and Numerical Study of Wind Turbine Seismic Behavior. Ph.D., University of California, San Diego, 309 pages; AAT 3449912.

Shyu, R. J., et. al. (2002). The characteristics of structural and ground vibration caused by the TRTS trains. Metro's Impact on Urban Living, Proc., World Metro Symp., Taipei, Taiwan, 610.

Sonmez, E. (2010). Deterministic and stochastic responses of smart variable stiffness and damping systems and smart tuned mass dampers. Ph.D., Rice University, 318 pages; AAT 3421186.

Soudkhah, M. (2010). Development of engineering continuum models for seismic soil-foundation interaction. Ph.D., University of Colorado at Boulder, 676 pages; AAT 3433347.

Trochides,A. (1991). Ground-borne vibrations in buildings near subways.Appl. Acoustics, 32, pp.289–296.

Turner, J. (2004). Vibration isolation: Harmonic and seismic forcing using the Wilson Theta method. ASHRAE Transactions: Symposia.

Ulusoy, H.S. (2011). Applications of System Identification in Structural Dynamics: A Realization Approach. Ph.D., University of California, Irvine, 119 pages; AAT 3444318.

Uzdin, A.M. et al. (2009). Performance analysis of seismic-insulating kinematic foundations on support elements with negative stiffness. Soil Mechanics and Foundation Engineering, Vol. 46, No. 3.

Weng, S. (2010). A new substructuring method for model updating of large-scale structures. Ph.D., Hong Kong Polytechnic University (Hong Kong), 319 pages; AAT 3448510.

Wieser, J. (2011). Assessment of Floor Accelerations in Yielding Buildings. M.Sc., University of Nevada, Reno, 272 pages; AAT 1494267.

Vadillo, E. G., Herreros, J., and Walker, J. G. (1996). Subjective reaction to structurally radiated sound from underground railways: Field studies. J. Sound Vib., 193(1), pp. 65–74.

Yang, Y.B. et al. (2008). Soil vibrations caused by underground moving trains. J. of Geotechnical and Geoenvironmental Engineering, ASCE.

Yao, M.M. (2010). Earthquake Wave-Soil-Structure Interaction Analysis of Tall Buildings. Ph.D., University of Victoria (Canada), 95 pages; AAT NR74121.

Yun, C. B. et al. (2000). Analytical frequency dependent infinite elements for soil- structure interaction analysis in two-dimensional medium. Eng. Struct., 22(3), pp. 258–271.

Zhong, P. (2011). Sensitivity of building response to variation in integration time step of response history analysis. M.S., University of California, Irvine, 70 pages; AAT 1490700.

Appendix

$m_F = \rho_F . V_F$, Foundation weight $= W_F = m_F . g$, Lorenz, H. (1955) calculated the weight of the accompanied vibrating soil with the foundation using the equation $W_S = f . A_F^{(4/3)} = [0.835].[a.b]^{(4/3)}$ ton, $m_S = W_S / g$ kg.

$$m_1 = m_F + m_S, \; J_1 = J_F + m_F . l_f^2 + J_S + m_S . l_S^2, \; l_F = \frac{m_S}{m_F} . l_S = \frac{m_S}{m_F}\left(\frac{d+h}{2} - l_F\right) = \frac{[m_S / m_F][(d+h)/2]}{[1 + (m_S / m_F)]}$$

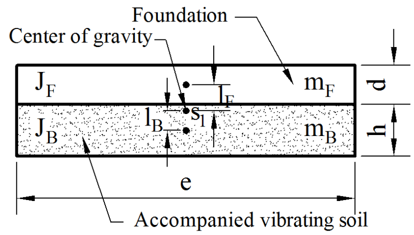

Fig. 4 Foundation with its accompanied vibrating toned sand

$h = \frac{W_S}{A_F . r_S}$, $l_S = [(d+h)/2 - l_F]$, $J_F = m_F.[(d^2 + e^2)/12]$, $J_S = m_S.[(h^2 + e^2)/12]$

Vertical embedding damping constant: the damping constant of radiation is $r_b = E_d / V_S$

Mass of the high tower: the density of high tower can be assumed as 1/10 of that of the foundation, i.e. $\rho_2 = \rho_1 / 10$

$m_2 = \rho_2.V_2$, $W_2 = m_2.g$, $J_2 = m_2.[(b^2 + c^2)/12]$