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WAVE REFLECTION AND BEACH PROFILE EVOLUTION UNDER SINUSOIDAL AND CNOIDAL LABORATORY WAVES

M. Shah Alam Khan[1]

ABSTRACT: Wave reflection and beach profile evolution characteristics are investigated with sinusoidal and cnoidal waves having the same specified wave height and period in a 90 ft-long wave tank at three nominal scales. Waveforms are not purely sinusoidal or cnoidal due to presence of waves reflected from the beach, and free secondary waves. Separated incident and reflected wave energy spectra detect energy at higher harmonics in addition to the fundamental frequency. Incident first harmonics are smaller in cnoidal tests than in sinusoidal tests. Reflected energy transfers from the fundamental to the first harmonic as the beach evolves, and transfers back to the fundamental only in cnoidal tests after establishment of the foreshore slope. Reflection coefficient, indicating relative magnitude of reflected energy, rises to a maximum early in the sinusoidal tests and changes slowly to equilibrium. Reflection coefficient changes consistently throughout the cnoidal tests. Sinusoidal waves change the profile in parts, mostly early in the test, whereas cnoidal waves change the profile steadily to equilibrium. Cnoidal waves create higher berm crest, deeper trough and steeper profile slope, and take longer than sinusoidal waves to reach equilibrium.

KEY WORDS: Cnoidal, sinusoidal, wave reflection, profile evolution.

INTRODUCTION

In most coastal hydrodynamic physical models near-shore waves are assumed to be sinusoidal in nature. This assumption simplifies interpretation of model results and allows laboratory waves to be generated with relatively less difficulty. Dean and Dalrymple (1984) and Hughes (1993) present the general first order wave generator solution for wave height to stroke ratio,

$$\frac{H}{S} = \frac{4\sinh kd}{\sinh 2kd + 2kd}\left\{\sinh kd + \frac{1-\cosh kd}{k(d+l)}\right\} \tag{1}$$

where H = wave height, S = wave paddle stroke, d = water depth, $k = 2\pi/L$ = wave number and L = wavelength; $l = 0$ for flap-type wave paddles hinged at the bottom and $l \to \infty$ for piston-type paddles. Goda (1967) identifies that periodic waves generated by a sinusoidally oscillating wave paddle include free secondary waves that propagate at a speed slower than the primary waves. The combined waveform changes spatially and

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[1] Institute of Water and Flood Management, BUET, Dhaka 1000, Bangladesh.

temporally in the wave tank. The waveform modification is more pronounced as the wave steepness, H/L, increases or relative depth, d/L, decreases. Dean and Dalrymple (1984) show that piston-type wave paddles more closely follow the water particle trajectories under shallow water waves and minimize generation of these 'free second harmonic' waves.

Cnoidal waves, relatively complex in theoretical formulation and difficult to generate in stable form in the laboratory, are believed to represent near-shore shallow water waves better than the sinusoidal waves. Based on theory developed by Korteweg and deVries (1895) for periodic long wave of permanent form in shallow water, Goring (1978) presents wave generation theory for cnoidal waves of permanent form and experimental verification in laboratory using a piston-type wave generator. The profile of cnoidal waves is given by,

$$\eta(x,t) = y_t - d + H\,\mathrm{cn}^2\left\{2K\left(\frac{x}{L}-\frac{t}{T}\right), m\right\} \qquad (2)$$

where, $\eta(x,t)$ = water surface profile, y_t = height of trough above horizontal bottom, cn = Jacobian elliptic cosine function, T = wave period, K(m) = complete elliptic integral of the first kind, and m = the elliptic modulus that defines the shape of wave. Cnoidal waves reduce to sinusoidal waves when $m \to 0$ and to solitary waves when $m \to 1$.

Cnoidal waves have more peaked crest, and flatter and longer trough than sinusoidal waves (Fig. 1). Consequently, energy dissipation, wave reflection and changes on the beach are different under sinusoidal and cnoidal waves. Since wave height is proportional to energy, this difference can be identified from beach profile evolution tests conducted with sinusoidal and cnoidal waves having the same wave height and period.

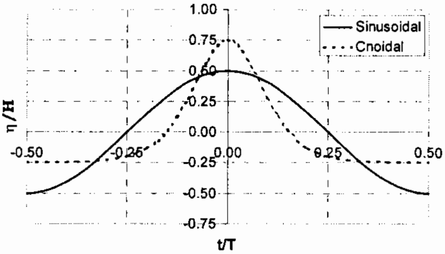

Fig. 1. Waveform of sinusoidal and cnoidal waves

LABORATORY TESTS

Experimental Set-up

Tests were conducted at three nominal scales in a 90 ft-long, 3 ft-wide and 2.5 ft-deep wave tank. Geometric similarity, deep water wave steepness (H_o/L_o), wave Froude number (H/gT^2), densimetric Froude number ($F_* = v_*^2/(\gamma' g D_{50})$), and particle Reynolds number ($R_* = v_* D_{50}/\nu$) were preserved by selecting the same sediment size and density, and the same fluid, where H_o = deep water wave height, L_o = deep water wave length, g = acceleration of gravity, v_* = bed shear velocity, D_{50} = median sediment grain diameter, γ' = relative unit weight of sediment = $(\rho_s - \rho)/\rho$, ρ = fluid density, ρ_s = sediment density, and ν = kinematic viscosity of fluid.

A piston-type wave generator having active wave absorption capability produced waves. Sinusoidal and cnoidal waves were generated from wave paddle displacement signals computed based on Eq.(1) and Eq.(2), respectively. Wave height was measured at four locations outside the offshore end of the beach using parallel-wire resistance gauges. The gauge separation distance, Δl, was constrained to: $0.05 < \Delta l/L < 0.45$ following a method developed by Goda (1985) for estimation of incident and reflected waves. Silica sand having a median grain diameter of 0.212 mm was used to construct the beach on a permeable frame so that percolation through the beach face simulated the natural groundwater.

Test Description

Two accretionary profile evolution tests were conducted at each scale: one with sinusoidal and one with cnoidal waves having the same wave height, both starting with the same, appropriately scaled, initial profile shape. Each 'test' was conducted in a sequence of several 'runs' until the equilibrium condition was attained. Typical run durations were 20, 30 or 60 minutes. The wave generator operated continuously during a run except in cnoidal wave tests where the wave generator was typically operated for 20 minutes at a stretch during a run. Beach profiles were measured with a semi-automatic profiler several times in a test during the interval between runs. Wave height, wave period and still water depth were determined from the nominal scale and preserved test properties. Table 1 summarizes the basic variables of the tests.

RESULTS AND DISCUSSION

Wave Characteristics

Waves generated in the laboratory are not purely sinusoidal or cnoidal. Changes in waveform occur due to presence of reflected waves from different parts of the beach and free second harmonic waves. Fig. 2(a) shows how the waveform changes with time during Test DSTi01. Changes are obvious as the initial foreshore slope is established at about 185 minutes into the test, and near the equilibrium at approximately 575 minutes. The specified wave period and wave height for these sinusoidal waves are 2.53 sec and 5.0 cm, respectively. Fig. 2(b) shows similar changes during Test DSTi02 conducted with cnoidal waves having the same specified wave height and period.

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Table 1. Summary of test variables

Test ID	Wave type	Nominal scale	Wave period, T (sec)	Wave height, H (cm)	Still water depth at wave paddle (cm)
DSTi01	Sinusoidal	0.85	2.53	5.00	33.10
DSTi02	Cnoidal	0.85	2.53	5.00	33.10
DSTi03	Sinusoidal	1.00	2.74	5.88	35.85
DSTi04	Cnoidal	1.00	2.74	5.88	35.85
DSTi05	Sinusoidal	0.77	2.41	4.55	27.70
DSTi06	Cnoidal	0.77	2.41	4.55	27.70

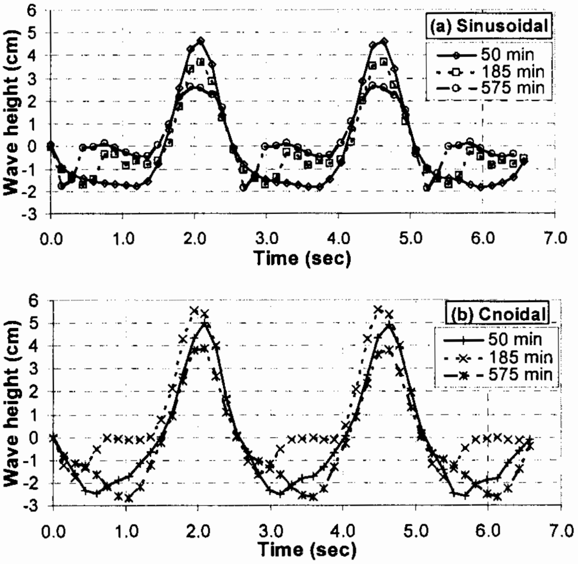

Fig. 2. Waveform modification in sinusoidal and cnoidal wave tests

The waves shown in Fig. 2 are combination of incident waves from the wave generator and reflected waves from the beach. The incident and reflected wave energy spectra are separated, as shown in Fig. 3, using wave record from two gauge-pairs and following Goda and Suzuki's (1976) procedure. Most incident waves are generated at the specified fundamental frequency of 0.395, while secondary wave components exist at other frequencies. Incident and reflected energy are present at the fundamental and first harmonic frequencies throughout both sinusoidal and cnoidal wave tests. Cnoidal incident first harmonics are not as large as the sinusoidal first harmonics.

From detailed spectral analysis, four harmonics are detected in the reflected wave spectra. The sinusoidal fundamental harmonic peaks at about 170 minute and gradually decreases to a stable value. Cnoidal reflected first harmonic decreases as the test progresses while the fundamental increases steadily. In both sinusoidal and cnoidal reflected wave spectra, energy transfers from the fundamental to the first harmonic as the beach evolves. In cnoidal wave spectra, energy from higher harmonics transfers back to the fundamental after the initial foreshore slope is established. In general, transfer of energy between harmonics is more pronounced in cnoidal tests.

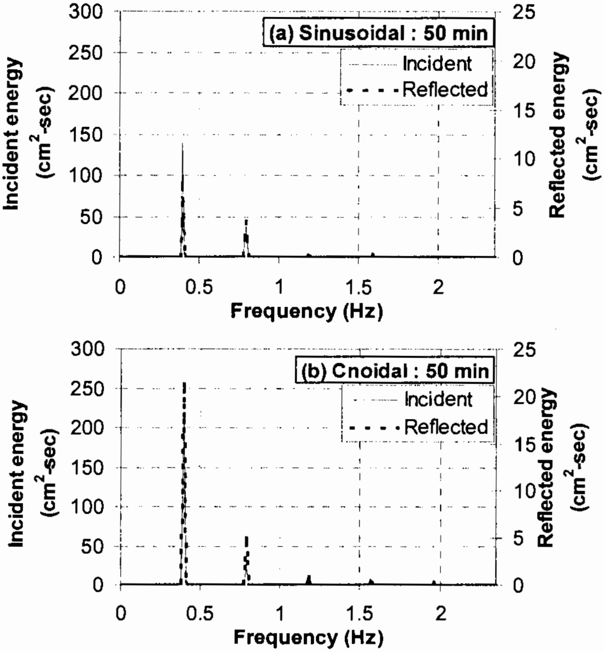

Fig. 3. Incident and reflected energy spectra

Most of the incident wave energy is dissipated on the beach by several processes including sediment transport and wave breaking, and the excess energy is reflected offshore from different parts of the beach. Wave reflection increases as the profile approaches an equilibrium condition under the wave action, and less energy is expended in changing the beach. The reflection coefficient, $C_R = H_r/H_i$, indicates the relative magnitude of wave reflection, where H_r = reflected wave height and H_i = incident wave height. Fig. 4 shows the change in C_R with profile evolution. In sinusoidal tests, C_R increases relatively fast and peaks early in the tests, which corresponds to establishment of the initial foreshore slope. In cnoidal tests, C_R increases to equilibrium consistently throughout the test. This indicates that sinusoidal waves cause most changes on the beach during the early part of the test whereas cnoidal waves change the beach through the entire duration of the test.

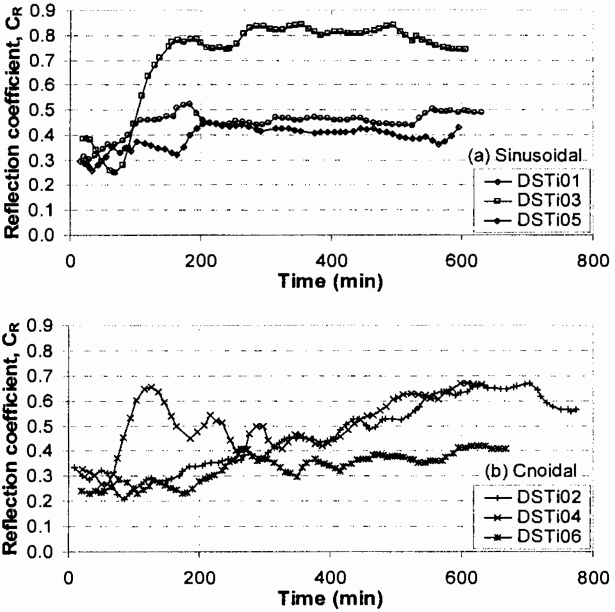

Fig. 4. Change in reflection coefficient with profile evolution

Profile Evolution to Equilibrium

Beach profiles evolve and approach equilibrium differently under sinusoidal and cnoidal waves. One way to specify equilibrium is when there is no longer any significant horizontal change in the still water line (SWL) location. Fig. 5 shows how the SWL moves with respect to the final SWL toward a mean equilibrium position where the fraction of total movement approaches zero at equilibrium. The SWL moves relatively fast to equilibrium position in sinusoidal tests whereas its fluctuation about the mean position is within a much wider range in cnoidal tests.

Equilibrium is also specified by the root mean square error (RMSE) of profile elevations between consecutive profiles. The RMSE is divided by the corresponding run duration so the average RMSE indicates the overall change of profile during the run. Fig. 6 shows how the average RMSE decays to equilibrium. Most changes occur during the initial part of the sinusoidal tests, and changes are relatively uniform toward the equilibrium. Significant changes occur throughout the cnoidal tests.

Sinusoidal waves change the profile 'in parts': the regions near the wave break point change first, then the berm crest grows and the foreshore slope develops. Cnoidal waves change the entire profile simultaneously throughout the test, and produce a higher berm crest, deeper trough and steeper overall profile slope than the sinusoidal waves. Profiles take longer to reach equilibrium under cnoidal waves.

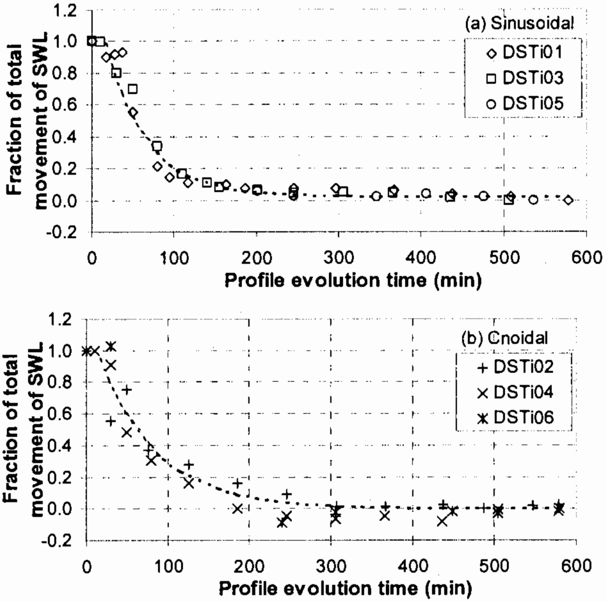

Fig. 5. Change in still water line (SWL) location

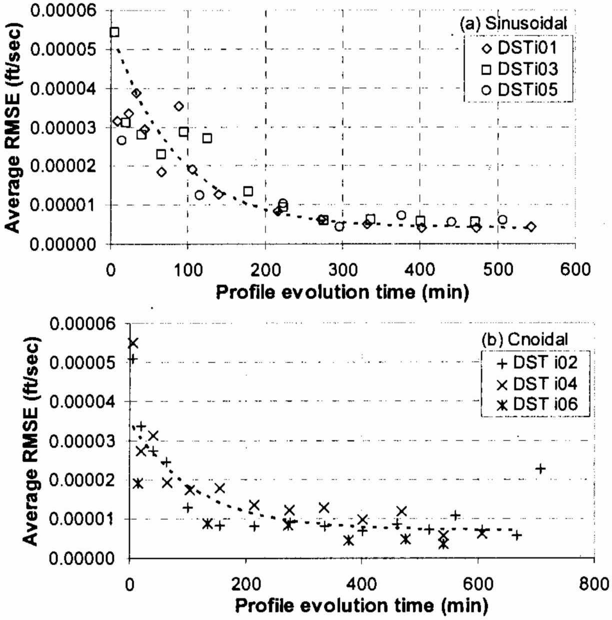

Fig. 6. Change in root mean square error (RMSE) between profiles

CONCLUSIONS

Accretionary profile evolution tests conducted in a wave tank show significant difference in profile evolution and wave reflection characteristics between sinusoidal and cnoidal waves. Waveform modification occurs in both sinusoidal and cnoidal waves due to presence of free secondary waves from the wave generator and reflected waves from the beach. Separation of incident and reflected wave energy spectra shows presence of higher harmonics, in addition to the specified fundamental, while incident and reflected energy are present at the fundamental and first harmonic throughout the tests. Reflected first harmonics are higher in sinusoidal tests. Energy transfers from the reflected fundamental to the first harmonic as the beach evolves in both sinusoidal and cnoidal tests. Reflected energy transfers back to the fundamental frequency after the foreshore slope is developed in cnoidal tests. Energy transfer between harmonics is more pronounced in cnoidal tests.

Reflection coefficient, indicating wave energy reflected from the beach, increases relatively fast and peaks early in sinusoidal tests whereas reflection increases consistently throughout the cnoidal tests. A constant, relatively high reflection coefficient denotes equilibrium condition. Movement of SWL and change in RMSE of profile elevations both indicate that in sinusoidal tests most profile changes occur early in the tests, and profile reaches equilibrium faster than cnoidal tests. Sinusoidal waves change the profile one region at a time whereas cnoidal waves simultaneously change the entire profile and create higher berm crest, deeper trough and steeper profile at equilibrium. Profile changes at equilibrium are less uniform in cnoidal tests.

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REFERENCES

Dean, R.G. and Dalrymple, R.A. (1984), Water Wave Mechanics for Engineers and Scientists, Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

Dean, R.G. and Dalrymple, R.A. (2002), Coastal Processes with Engineering Applications, Cambridge Univ. Press, Cambridge.

Goda, Y. (1967), "Travelling secondary wave crests in wave channels", In Laboratory Investigation on Wave Transmission Over Breakwaters, Rept. No. 13, 32-38, Port and Harbor Research Institute, USA.

Goda, Y. (1985), Random Seas and Design of Maritime Structures, University of Tokyo Press, Japan.

Goda, Y. and Suzuki, Y. (1976), "Estimation of incident and reflected waves", Proceedings of 15th Coastal Eng. Conf., ASCE, Vol. 1, 828-845.

Goring, D.G. (1978), Tsunamis – the Propagation of Long Waves onto a Shelf, Rept. KH-R-38, W.M. Keck Laboratory of Hydraulics and Water Resources, California Institute of Technology, Pasadena.

Hughes, S.A. (1993), Physical Models and Laboratory Techniques in Coastal Engineering, World Scientific, Singapore.

Korteweg, D.J. and de Vries, G. (1895), "On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves", Philosophy Magazine, Series 5, Vol. 39, 422-443.

NOTATIONS

cn = Jacobian elliptic cosine function
C_R = Reflection coefficient
d = Water depth
D_{50} = Median sediment grain diameter
F_* = Densimetric Froude number
g = Acceleration of gravity
H = Wave height at a specified depth
H_o = Deep water wave height
H_i = Incident wave height
H_r = Reflected wave height
k = Wave number
K = Complete elliptic integral of the first kind
l = Length variable for wave paddle type
L = Wavelength
L_o = Deep water wave length
m = Elliptic modulus
R_* = Particle Reynolds number
S = Wave paddle stroke
t = Time
T = Wave period
v_* = Bed shear velocity
x = Horizontal distance
y_t = Height of trough above horizontal bottom
Δl = Gauge separation distance
γ' = Relative unit weight of sediment
η = Water surface elevation
ν = Kinematic viscosity of fluid
ρ = Density of fluid
ρ_s = Density of sediment

ABBREVIATIONS

RMSE = Root Mean Square Error
SWL = Still Water Line